J. Fluid Mech. (1992), vol. 238, pp. 297–323
Printed in Great Britain

On a class of unsteady three-dimensional Navier–Stokes solutions relevant to rotating disc flows: threshold amplitudes and finite-time singularities

By PHILIP HALL,[1] P. BALAKUMAR[2] AND D. PAPAGEORGIU[3]

[1] Mathematics Department, University of Manchester, Oxford Road, M139TL, UK

[2] High Technology Corporation, Hampton, VA, USA

[3] New Jersey Institute of Technology, Newark, NJ, USA

(Received 28 November 1990 and in revised form 25 October 1991)

A class of 'exact' steady and unsteady solutions of the Navier–Stokes equations in cylindrical polar coordinates is given. The flows correspond to the motion induced by an infinite disc rotating in the (x, y)-plane with constant angular velocity about the z-axis in a fluid occupying a semi-infinite region which, at large distances from the disc, has velocity field proportional to $(x, -y, 0)$ with respect to a Cartesian coordinate system. It is shown that when the rate of rotation is large Kármán's exact solution for a disc rotating in an otherwise motionless fluid is recovered. In the limit of zero rotation rate a particular form of Howarth's exact solution for three-dimensional stagnation-point flow is obtained. The unsteady form of the partial differential system describing this class of flow may be generalized to time-periodic flows. In addition the unsteady equations are shown to describe a strongly nonlinear instability of Kármán's rotating disc flow. It is shown that sufficiently large perturbations lead to a finite-time breakdown of that flow whilst smaller disturbances decay to zero. If the stagnation point flow at infinity is sufficiently strong the steady basic states become linearly unstable. In fact there is then a continuous spectrum of unstable eigenvalues of the stability equations but, if the initial-value problem is considered, it is found that, at large values of time, the continuous spectrum leads to a velocity field growing exponentially in time with an amplitude decaying algebraically in time.

1. Introduction

Our concern is with the strongly nonlinear instability of the boundary layer on a rotating disc, an important feature of our investigation is that the stability problem which we discuss corresponds to an exact solution of the Navier–Stokes equations. In addition, our investigation uncovers a class of exact steady and unsteady Navier–Stokes solutions relevant to the flow over a rotating disc immersed in a three-dimensional stagnation-point flow field.

The importance of the problem for the boundary layer on a rotating disc is due to the fact that the flow can be thought of as a prototype stability problem for boundary-layer flows over swept wings. Some years ago Gregory, Stuart & Walker (1955) showed that the flow over a rotating disc is highly unstable to an inviscid 'crossflow' instability associated with the highly inflexional velocity profiles which occur in certain directions. Their calculations pointed to the particular importance of a stationary mode of instability associated with an effective velocity profile having an inflexion point at a position of zero flow velocity. The structure of the Gregory *et al.* mode in the nonlinear case was later discussed by Bassom & Gajjar (1988), the latter authors show that in that regime a nonlinear critical-layer structure develops. This mode is apparently the one most preferred in an experimental situation if the background disturbance level is sufficiently small. However, though most experimental investigations of the rotating-disc problem have clearly identified the stationary crossflow structure described by Gregory *et al.* (1955), some experiments have pointed to the existence of a second type of vortex structure associated with some type of subcritical response caused by another type of instability, see for example Federvov *et al.* (1976).

A possible cause for this second type of stationary vortex structure is the viscous stationary crossflow vortex identified numerically by Malik (1986) and described using essentially triple-deck theory by Hall (1986). Later MacKerrell (1987) was able to show that this mode is destabilized by nonlinear effects and therefore might cause the subcritical instability observed experimentally. However, a key feature of the mechanism described by Hall (1986) is that the crucial balance of forces leading to instability is one between Coriolis and viscous forces, thus in swept-wing flows this mechanism is possibly not operational. An alternative source of breakdown caused by a finite-amplitude instability in more general three-dimensional boundary layers is the one discussed in this paper. Though we shall formulate and solve the resulting nonlinear interaction equations in cylindrical polar coordinates it is easy to see the relevance of the structures we find to flows more naturally described in Cartesian coordinates.

In §2 we shall formulate the nonlinear interaction equations describing the flow over a rotating disc immersed in a three-dimensional stagnation point flow. In §3 we shall discuss some steady equilibrium states of these equations, in particular we describe the non-unique nature of the solutions of these equations. In §4 we concentrate on the unsteady form of the interaction equations and we discuss the linear and nonlinear instability of the flow over a rotating disc in an otherwise still fluid. Our calculations in that section point clearly to a threshold amplitude response of the flow; thus a sufficiently large initial disturbance causes an unbounded velocity field to develop after a finite time. The singularity structure associated with this 'blow-up' is discussed in §5. Since the interaction equations which we derive in §3 are obtained without neglecting any terms in the Navier–Stokes equations the singularity discussed in §5 is a singularity of the full Navier–Stokes equations in three dimensions. Singularities of the Euler equations have been discussed by, for example, Stuart (1987), however, there is no obvious connection between that work and that discussed here. Finally, in §6 we draw some conclusions.

2. The equations for combined disc–stagnation point flows

With respect to cylindrical polar coordinates (r, θ, z) the Navier–Stokes equations may be written

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}+\begin{pmatrix} -\dfrac{v^2}{r} \\ \dfrac{uv}{r} \\ 0 \end{pmatrix}=\frac{-1}{\rho}\boldsymbol{\nabla} p+\nu\Delta\boldsymbol{u}+\nu\begin{pmatrix} -\dfrac{u}{r^2}-\dfrac{2}{r^2}v_\theta \\ -\dfrac{v}{r^2}+\dfrac{2u_\theta}{r^2} \\ 0 \end{pmatrix}, \qquad (2.1\,a)$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.1b}$$

where (u, v, w) is the velocity field corresponding to (r, θ, z) and p, ρ and ν are the fluid pressure, density and kinematic viscosity respectively. The operators $\boldsymbol{\nabla}$ and Δ appearing in (2.1) are the gradient and Laplacian operators in cylindrical polar coordinates. We define dimensionless time and axial variables T and ζ by

$$T = \Omega t, \quad \zeta = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} z, \tag{2.2a, b}$$

where Ω is a constant angular velocity. We seek a solution of (2.1) in the form

$$u = \Omega r \bar{u}(\zeta, T) + \tfrac{1}{2} r\{\Omega U(\zeta, T)\, \mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}\}, \tag{2.3a}$$

$$v = \Omega r \bar{v}(\zeta, T) + \tfrac{1}{2} r\{\mathrm{i}\Omega U(\zeta, T)\, \mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}\}, \tag{2.3b}$$

$$w = (\Omega \nu)^{\frac{1}{2}}\, \bar{w}(\zeta, T), \tag{2.3c}$$

$$\frac{p}{\rho} = \tfrac{1}{2}\lambda \Omega^2 r^2 + \nu \Omega \bar{p}(\zeta, T) + \Omega^2 r^2 \{J(T)\, \mathrm{e}^{2\mathrm{i}\theta} + \text{cc.}\}. \tag{2.3d}$$

Here λ is a constant whilst c.c. denotes 'complex' conjugate'. We note that (2.3) reduces to Kármán's solutions for the flow over a rotating disc if we set $U = J = 0$. If we substitute (2.3) into (2.1) we find the crucial result that the nonlinear terms in the radial and azimuthal momentum equations generate no terms proportional to $\mathrm{e}^{\pm 4\mathrm{i}\theta}$. This means that (2.3) is an exact Navier–Stokes solution and we find that the equations to determine the functions appearing in (2.3) are

$$\bar{u}_T + \bar{u}^2 + |U|^2 - \bar{v}^2 + \bar{w}\bar{u}_\zeta = -\lambda + \bar{u}_{\zeta\zeta}, \tag{2.4a}$$

$$\bar{v}_T + 2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta = \bar{v}_{\zeta\zeta}, \tag{2.4b}$$

$$\bar{w}_T + \bar{w}\bar{w}_\zeta = -\bar{p}_\zeta + \bar{w}_{\zeta\zeta}, \tag{2.4c}$$

$$2\bar{u} + \bar{w}_\zeta = 0, \tag{2.4d}$$

and

$$U_T + 2\bar{u}U + \bar{w}U_\zeta = -J + U_{\zeta\zeta}. \tag{2.5}$$

Before writing down boundary conditions appropriate to (2.4), (2.5) it is perhaps worth commenting on the motivation for the choice of the special form (2.3). Balakumar, Hall & Malik (1991) investigated the instability of Kármán's solution to non-parallel travelling wave modes of wavenumber n in the azimuthal direction. These high-Reynolds-number modes can be made nonlinear in the manner suggested by the vortex–wave interaction structure of Hall & Smith (1991). In that structure the amplitude of the non-parallel mode with azimuthal wavenumber n is adjusted until it drives a mean flow correction comparable with the unperturbed state. For $O(1)$ values of n it turns out that, because of the comparable size of the three disturbance velocity components in the analysis of Balakumar *et al.* (1991), only the $n = \pm 2$ modes can be made strongly nonlinear in the manner described by Hall & Smith (1991) and then the appropriate form of the disturbed flow is (2.3). However, the structure (2.3), suggested by the interaction described by Hall & Smith, is applicable at all Reynolds numbers rather than just at high Reynolds numbers where the work of Balakumar *et al.* (1991) applies. Thus we can interpret (2.3) as a 'mean field' type of disturbed flow with the θ-dependent part representing a wave superimposed on Kármán's solution which now evolves in time as the disturbance develops. We note that in this interaction no terms in the Navier–Stokes equations have been neglected.

We close this section with a discussion of the boundary conditions associated with (2.4), (2.5). We assume that as the flow evolves, the mean (with respect to θ) part of the velocity field, i.e. $(\bar{u}, \bar{v}, \bar{w})$, satisfies

$$\bar{u} = 0, \quad \bar{v} = 1, \quad \bar{w} = 0, \quad \zeta = 0, \tag{2.6a}$$

$$\bar{u}, \bar{v} \to 0, \quad \zeta \to \infty. \tag{2.6b}$$

Next we assume that the wavelike part of the flow satisfies

$$U = 0, \quad \zeta = 0, \quad U \to \gamma \mathrm{e}^{\mathrm{i}NT}, \quad \zeta \to \infty, \tag{2.7}$$

where γ is a constant and N is a constant dimensionless frequency. In order that the $\bar{u}$, and U equations are consistent with the above conditions λ and J must be chosen such that

$$\lambda = -\gamma^2, \quad J = -\mathrm{i}\gamma N \mathrm{e}^{\mathrm{i}NT}. \tag{2.8a, b}$$

Having made the above choice of boundary conditions we can seek solutions of (2.4), (2.5) which are periodic in time with period $2\pi/N$, the steady-state solutions of (2.4), (2.5) are found by setting $N = 0$. It follows from the form of the nonlinear term U^2 in (2.4a) that we can seek solutions $(\bar{u}, \bar{v}, \bar{w})$ independent of time so that $(\bar{u}, \bar{v}, \bar{w})$ and U satisfy

$$\mathrm{i}N\bar{U}^* + 2\bar{u}\bar{U}^* + \bar{w}\bar{U}^*_\zeta = \bar{U}^*_{\zeta\zeta} + \mathrm{i}N\gamma, \tag{2.9a}$$

$$\bar{u}^2 + |\bar{U}^*|^2 - \bar{v}^2 + \bar{w}\bar{u}_\zeta = \gamma^2 + \bar{u}_{\zeta\zeta}, \tag{2.9b}$$

$$2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta = \bar{v}_{\zeta\zeta}, \tag{2.9c}$$

$$2\bar{u} + \bar{w}_\zeta = 0, \tag{2.9d}$$

$$\bar{u} = 0, \quad \bar{v} = 1, \quad \bar{w} = 0, \quad \bar{U}^* = 0, \quad \zeta = 0, \tag{2.9e}$$

$$\bar{u}, \bar{v} \to 0, \quad \zeta \to \infty, \quad \bar{U}^* \to \gamma, \quad \zeta \to \infty, \tag{2.9f}$$

where we have replaced $U(\zeta, T)$ by $\mathrm{e}^{\mathrm{i}NT}\bar{U}^*(\zeta)$. The non-periodic solutions satisfy

$$U_T + 2\bar{u}U + \bar{w}U_\zeta = U_{\zeta\zeta} + \mathrm{i}N\gamma\,\mathrm{e}^{\mathrm{i}NT}, \tag{2.10a}$$

$$\bar{u}_T + \bar{u}^2 + |U|^2 - \bar{v}^2 + \bar{w}\bar{u}_\zeta = \gamma^2 + \bar{u}_{\zeta\zeta}, \tag{2.10b}$$

$$\bar{v}_t + 2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta = \bar{v}_{\zeta\zeta}, \tag{2.10c}$$

$$2\bar{u} + \bar{w}_\zeta = 0, \tag{2.10d}$$

$$\bar{u} = 0, \quad \bar{v} = 1, \quad \bar{w} = 0, \quad U = 0, \quad \zeta = 0, \tag{2.10e}$$

$$\bar{u} = \bar{v} = 0, \quad \zeta = 0, \quad U \to \gamma \mathrm{e}^{\mathrm{i}NT}, \quad \zeta \to \infty, \tag{2.10f}$$

$$\bar{u} = \hat{u}(\zeta), \quad \bar{v} = \hat{v}(\zeta), \quad U = \hat{U}(\zeta), \quad T = 0 \tag{2.10g}$$

Thus the periodic solutions can be found by integrating an ordinary differential system, (2.9), whereas the unsteady modes satisfy a parabolic partial differential system. For that reason we have been required in (2.10) to give initial conditions to completely specify the problem for $\bar{u}, \bar{v}, \bar{w}$ and U. Furthermore, we note that (2.10) can be regarded as the appropriate nonlinear initial-value instability problem for the periodic problem (2.9). In the next section we shall discuss the solution of (2.9).

3. Steady solutions of the interaction equations

In order to begin the solution of (2.9) by some appropriate numerical method it is convenient to discuss limiting forms of that system which can then be used to begin the calculation. The first limit we consider is $\gamma \rightarrow 0$ in which case the problem for $\bar{u}$, $\bar{v}$ and $\bar{w}$ becomes uncoupled from that for $\bar{U}^*$ and is in fact simply Kármán's solution. Thus we know that in the limit $\gamma \rightarrow 0$, $\bar{u}'(0) \approx 0.50$, $\bar{v}'(0) \approx -0.61$. Another known flow is found in the limit $\gamma \rightarrow \infty$. In that limit we write $N = \gamma N_0$ and expand the velocity field as

$$\bar{u} = \gamma \bar{u}_0 + \dots, \tag{3.1a}$$

$$\bar{v} = \bar{v}_0 + \dots, \tag{3.1b}$$

$$\bar{w} = \gamma^{\frac{1}{2}} \bar{w}_0 + \dots, \tag{3.1c}$$

$$\bar{U}^* = \gamma U_0 + \dots, \tag{3.1d}$$

where the functions appearing in the above expansions are functions of the stretched variable $\xi = \gamma^{\frac{1}{2}} \zeta$. In this limit the problem for $\bar{u}_0, U_0, \bar{w}_0$ decouples from $\bar{v}_0$ and we obtain the system

$$\mathrm{i} N_0 U_0 + 2\bar{u}_0 U_0 + \bar{w}_0 U_0' = U_0'' + \mathrm{i} N_0, \tag{3.2a}$$

$$\bar{u}_0^2 + |U_0|^2 + \bar{w}_0 \bar{u}_0' = 1 + \bar{u}_0'', \tag{3.2b}$$

$$2\bar{u}_0 + \bar{w}_0' = 0, \tag{3.2c}$$

$$\bar{u}_0 = 0, \quad \bar{w}_0 = 0, \quad U_0 = 0, \quad \xi = 0, \tag{3.2d}$$

$$\bar{u}_0 \rightarrow 0, \quad U_0 \rightarrow 1, \quad \xi \rightarrow \infty, \tag{3.2e}$$

In the special case $N_0 = 0$ we can relate (3.2) to a special case of the three-dimensional stagnation-point flows considered by Howarth (1951), Davey (1961), Banks & Zaturska (1989). We recall that, with respect to Cartesian coordinate x, y, z, Howarth identified the class of exact Navier–Stokes solutions given by

$$u = \frac{U_e x}{l} f'(\eta), \quad v = \frac{V_e y}{l} g'(\eta), \quad w = -\left(\frac{\nu}{l U_e}\right)^{\frac{1}{2}} (U_e f(\eta) + V_e g(\eta)), \tag{3.3}$$

where U_e, V_e are velocities and l is a length. The variable η is defined by

$$\eta = \left(\frac{U_e}{\gamma l}\right)^{\frac{1}{2}} z. \tag{3.4}$$

Here f', g' satisfy

$$f'^2 - (f + \alpha g) f'' = 1 + f''', \tag{3.5a}$$

$$g'^2 - \left(g + \frac{1}{\alpha} f\right) g'' = 1 + \frac{1}{\alpha} g''', \tag{3.5b}$$

with $\alpha = V_e / U_e$ and subject to

$$f = g = f' = g' = 0, \quad \eta = 0, \tag{3.6a}$$

$$f', g' \rightarrow 1, \quad \eta \rightarrow \infty. \tag{3.6b}$$

These solutions correspond to a three-dimensional stagnation-point flow above the plane $z = 0$. In the special case $\alpha = -1$ we can relate f' and g' to the functions $\bar{u}_0$, U_0 with $N_0 = 0$ by writing

$$2\bar{u}_0 = f' - g', \quad -\bar{w}_0 = f - g, \quad 2U_0 = f' + g'.$$

For this value of α Davey (1961) gives $f'' = 1.2729$, $g'' = -0.8112$ which suggests that for large γ the solutions of (2.9) with $N_0 = 0$ are such that

$$\bar{u}'(0) = 1.042\gamma^{\frac{3}{2}} + \ldots, \tag{3.7a}$$

$$\bar{U}^{*\prime}(0) = 0.231\gamma^{\frac{3}{2}} + \ldots. \tag{3.7b}$$

However, we shall see below that the solution of (2.9) is not unique so Davey's solution corresponds to only one of our solutions at large values of γ. For non-zero values of N_0 a similar asymptotic structure can be obtained but the coefficients in the expansions (3.7 a, b) will, of course, be functions of the frequency. Before giving the results of our numerical investigation of (2.9) we note that, from (3.1), for $\gamma \gg 1$, the dominant terms in the steady-state solution of (2.3 a, b, c) are such that

$$(u, v, w) \sim r\Omega\gamma\left(\bar{u}_0 + U_0\cos 2\theta,\ -U_0\sin 2\theta, \left(\frac{\nu}{\Omega\gamma}\right)^{\frac{1}{2}} w_0\right), \tag{3.8}$$

where without any loss of generality we have taken U_0 to be real. If we transform (3.8) to Cartesian coordinates we obtain a velocity field

$$\boldsymbol{u} \sim \Omega\gamma\left[x\bar{u}_0 + xU_0,\quad y\bar{u}_0 - yU_0,\quad \left(\frac{\nu}{\Omega\gamma}\right)^{\frac{1}{2}}\bar{w}_0\right], \tag{3.9}$$

and comparison of (3.3), (3.9) then confirms our previous remarks.

In practice the numerical solution of (2.9), and indeed the reduced large γ problem, (3.2), is not straightforward. The reason why there is a difficulty with the numerical solution of (3.2) was first discussed by Davey (1961) and later in more detail by Schofield & Davey (1967). In order to see what this difficulty is we consider the large ζ limit of the equations to determine $(\bar{u}, \bar{v}, \bar{w}, \bar{U}^*)$ in (2.9) with $N = 0$. Suppose that for $\zeta \gg 1$ we write

$$(\bar{u}, \bar{v}, \bar{w}, \bar{U}^*) = (u^+, v^+, w^+ + w_\infty,\quad \gamma + U^+),$$

where w_∞ is a constant and u^+, v^+, w^+ and U^+ are small. For simplicity we assume that U^+ is real. It is an easy matter to show that the linearized equations for u^+, v^+ can be reduced to

$$\left(\frac{\partial^2}{\partial\zeta^2} - w_\infty\frac{\partial}{\partial\zeta}\right)^2 u^+ = 4\gamma^2 u^+, \tag{3.10a}$$

$$\left(\frac{\partial^2}{\partial\zeta^2} - w_\infty\frac{\partial}{\partial\zeta}\right) v^+ = 0, \tag{3.10b}$$

and U^+, w^+ can be found in terms of u^+, v^+. Thus for large ζ we can write

$$u^+ = A_1\exp(m_1\zeta) + A_2\exp(m_2\zeta) + A_3\exp(m_3\zeta),$$

$$v^+ = B_1\exp(w_\infty\zeta).$$

Here m_1, m_2, m_3 are given by the values of m which satisfy

$$2m = w_\infty \pm (w_\infty^2 \pm 8\gamma)^{\frac{1}{2}} \quad (m_r < 0), \tag{3.11}$$

and we note that two of the required values will be complex when $|\gamma > \frac{1}{8}w_\infty^2$. Thus we have five independent constants w_∞, A_1, A_2, A_3 and B_1 which can be iterated upon in order to satisfy the four required conditions at $\zeta = 0$. It follows that there will be a continuum of solutions of (2.9) since there is no reason to ignore any of the decaying

γ	w_{∞}	$\bar{u}'(0)$	$\bar{v}'(0)$	$\bar{U}^{*\prime}(0)$
0.02	−0.355	0.511 528 74	−0.609 315 13	0.012 817 56
0.02	−0.390	0.510 899 63	−0.611 512 23	0.010 152 61
0.02	−0.410	0.510 801 89	−0.612 081 08	0.009 453 87
0.02	−0.450	0.510 692 53	−0.612 916 48	0.008 422 26
0.02	−0.500	0.510 622 90	−0.613 669 51	0.007 487 87
0.02	−0.550	0.510 585 52	−0.614 242 23	0.006 774 97
0.02	−0.600	0.510 564 18	−0.614 700 76	0.006 203 02
0.02	−0.650	0.510 551 85	−0.615 079 74	0.005 729 62
0.02	−0.700	0.510 544 93	−0.615 400 01	0.005 329 12
0.02	−0.750	0.510 534	−0.615 675 22	0.004 984 69
0.02	−0.800	0.510 540 15	−0.615 383	0.004 684 59
0.02	−0.850	0.510 540 34	−0.616 125 71	0.004 420 34
0.02	−0.900	0.510 541 51	−0.616 312 96	0.004 185 57
0.02	−0.950	0.510 543 34	−0.616 480 51	0.003 975 42
0.02	−1.000	0.510 545 60	−0.616 631 43	0.003 786 07
0.02	−1.050	0.510 548 15	−0.616 768 16	0.003 614 47
0.02	−1.100	0.510 550 88	−0.616 892 65	0.003 458 17
0.02	−1.150	0.510 553 70	−0.617 006 54	0.003 315 16
0.02	−1.200	0.510 556 57	−0.617 111 15	0.003 183 77
0.02	−1.250	0.510 559 45	−0.617 207 60	0.003 062 61
0.02	−1.300	0.510 562 31	−0.617 296 82	0.002 950 50
0.02	−1.350	0.510 565 13	−0.617 379 62	0.002 846 46
0.02	−1.400	0.510 567 90	−0.617 456 66	0.002 749 62
0.02	−1.500	0.510 573 24	−0.617 595 80	0.002 574 72
0.02	−1.600	0.510 578 30	−0.617 718 05	0.002 421 00
0.02	−1.700	0.510 583 06	−0.617 826 35	0.002 284 79
0.02	−1.800	0.510 587 53	−0.617 922 99	0.002 163 23
0.1	−0.510	0.520 252 70	−0.606 635 20	0.039 342 70
0.1	−0.550	0.518 461 72	−0.612 095 94	0.033 826 40
0.1	−0.600	0.517 961 37	−0.615 119 10	0.030 743 44
0.1	−0.650	0.517 791 86	−0.617 256 74	0.028 527 67
0.1	−0.700	0.517 751 62	−0.618 953 71	0.026 745 43
0.1	−0.750	0.517 774 57	−0.620 370 81	0.025 241 72
0.1	−0.800	0.517 831 40	−0.621 589 02	0.023 938 45
0.1	−0.850	0.517 907 06	−0.622 656 39	0.022 788 91
0.1	−0.900	0.517 993 09	−0.623 604 49	0.021 762 19
0.1	−0.950	0.518 084 49	−0.624 455 42	0.020 836 38
0.1	−1.000	0.518 178 18	−0.625 225 48	0.019 995 20
0.1	−1.100	0.518 365 38	−0.626 569 93	0.018 519 29
0.1	−1.200	0.518 546 17	−0.627 709 15	0.017 261 78
0.1	−1.300	0.518 717 23	−0.628 690 11	0.016 174 16
0.1	−1.400	0.518 877 48	−0.629 545 60	0.015 222 17
0.1	−1.500	0.519 026 93	−0.630 299 43	0.014 380 70
0.1	−1.600	0.519 166 05	−0.630 969 49	0.013 630 75
0.1	−1.700	0.519 295 52	−0.631 569 52	0.012 957 62
0.1	−1.800	0.519 416 06	−0.632 110 33	0.012 349 70
0.1	−1.900	0.519 528 43	−0.632 600 51	0.011 797 69
0.5	−0.600	0.681 978 09	−0.662 317 75	0.201 292 54
0.5	−0.800	0.688 317 33	−0.713 905 32	0.162 518 68
0.5	−1.000	0.697 557 16	−0.733 101 49	0.144 691 29
0.5	−1.500	0.716 533 43	−0.759 888 76	0.115 976 86
0.5	−2.000	0.730 267 10	−0.775 524 48	0.097 448 03
0.5	−2.500	0.740 550 61	−0.786 144 33	0.084 206 84
0.5	−3.000	0.748 526 20	−0.793 908 66	0.074 206 19
0.5	−3.500	0.754 890 53	−0.799 859 08	0.066 363 31
0.5	−4000	0.760 086 93	−0.804 575 55	0.060 003 20
0.5	−4500	0.764 410 04	−0.808 410 92	0.054 824 46
0.5	−5000	0.768 063 12	−0.811 593 60	0.050 540 39

TABLE 1. Values of $\bar{u}'(0)$, $\bar{v}'(0)$ and $\bar{U}^{*\prime}(0)$ obtained for different values of γ and w_{∞}

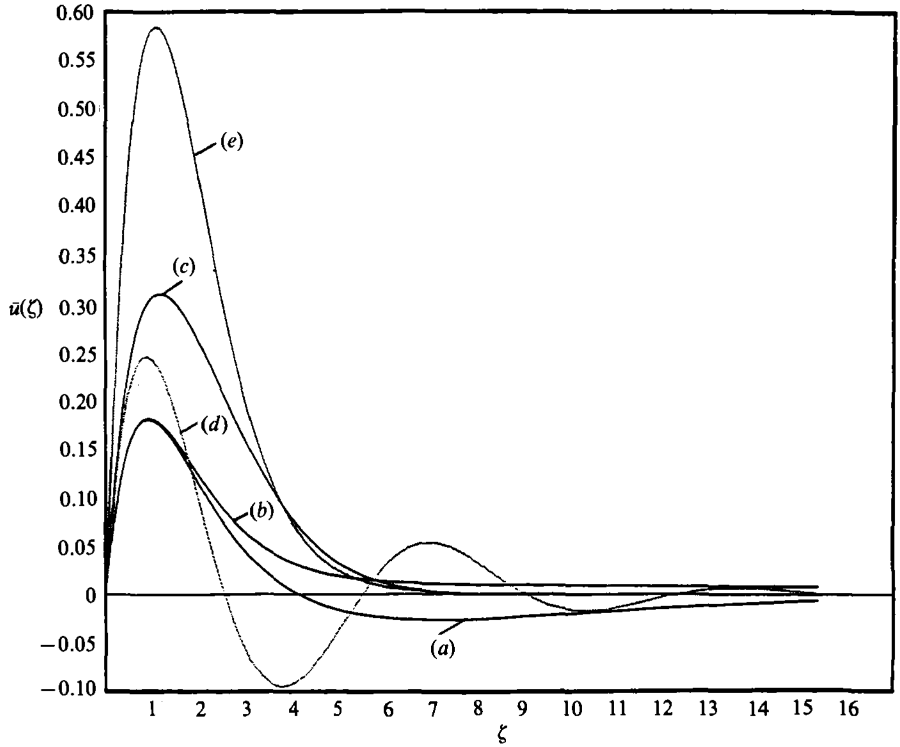

FIGURE 1. The function $\bar{u}(\zeta)$ for different values of γ, w_∞, (*a*) $\gamma = 0.02$, $w_\infty = -0.355$; (*b*) $\gamma = 0.02$, $w_\infty = -1.8$; (*c*) $\gamma = 0.5$, $w_\infty = -1.8$; (*d*) $\gamma = 0.5$, $w_\infty = -0.6$; (*e*) $\gamma = 1$, $w_\infty = -2.828$.

exponential solutions. Clearly w_∞ will vary in this continuum so the solutions can be labelled by the size of the inflow towards the disc at infinity. This situation persists in the large γ limit where $w_\infty^2 \sim \gamma$; Schofield & Davey (1967) argued that the solutions should in this case be fixed by discarding the slowest decaying exponential. Whilst it is certainly true that this fixes the solution, there is no basis for making such an assumption. Having made that assumption Schofield & Davey concluded that, in our notation, $w_\infty^2 = 8\gamma$. Interestingly enough we shall see later that this choice of w_∞ fixes the boundary between linearly stable and unstable solutions of (2.9). We now present results obtained for the system (2.9) in the steady case $N = 0$.

As mentioned above, at large values of ζ we have five constants, $A_1, B_1, A_2, A_3, w_\infty$ at our disposal once the constant γ has been fixed. In our calculations we fixed γ, w_∞ and iterated on the remaining four constants until the required boundary conditions at $\zeta = 0$ were satisfied after integrating the differential equations in (2.9) from a suitably large value of ζ to the origin. This integration was carried out using a fourth-order Runge–Kutta scheme or a compact finite-difference scheme. We concentrated our attention on the cases $\gamma = 0.02$, 0.1, 0.5 and in table 1 we show the values of $\bar{u}'(0)$, $\bar{v}'(0)$, $\bar{U}^{*\prime}(0)$ obtained for the different values of γ, w_∞ shown. We see that for each of the values of γ there are values of w_∞^2 greater and less than 8γ. For each of the values of γ used, we were able to find solutions of (2.9) only for w_∞ less than some critical value. Thus for example when $\gamma = 0.02$ we were unable to find solutions of (2.9) for w_∞ greater than -0.355. Our calculations suggested that this minimum value decreases when γ increases. In figures 1–4 we show the functions $\bar{u}, \bar{v}, \bar{w}, \bar{U}^*$ for different values of γ, w_∞. Figures 1–4 show that $\bar{u}, \bar{w}, \bar{U}^*$ are not always monotonic

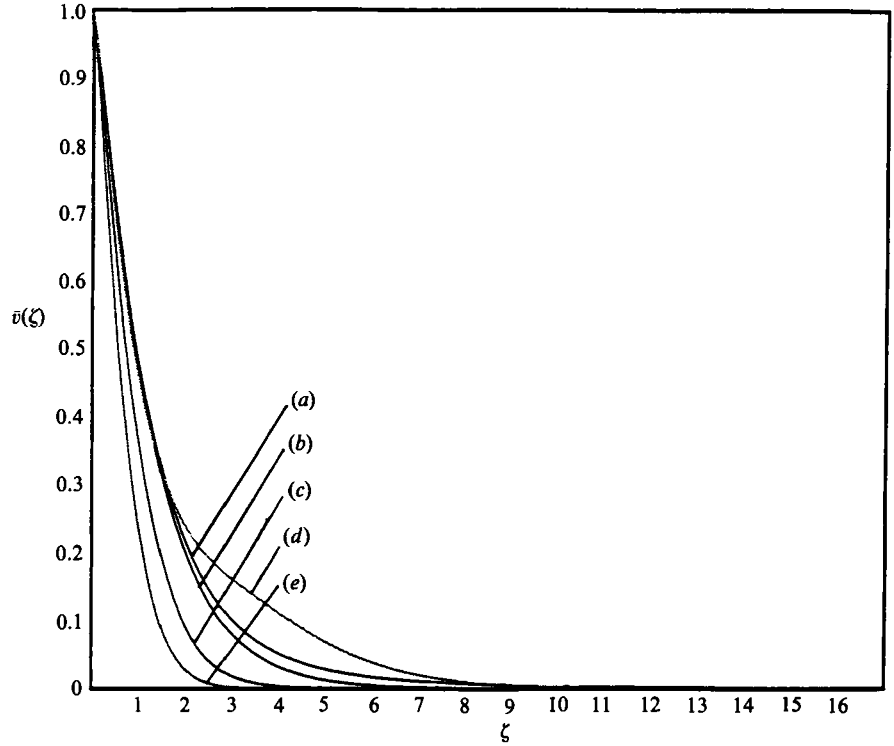

FIGURE 2. The function $\bar{v}(\zeta)$ for different values of γ, w_∞. (see figure 1 for values.)

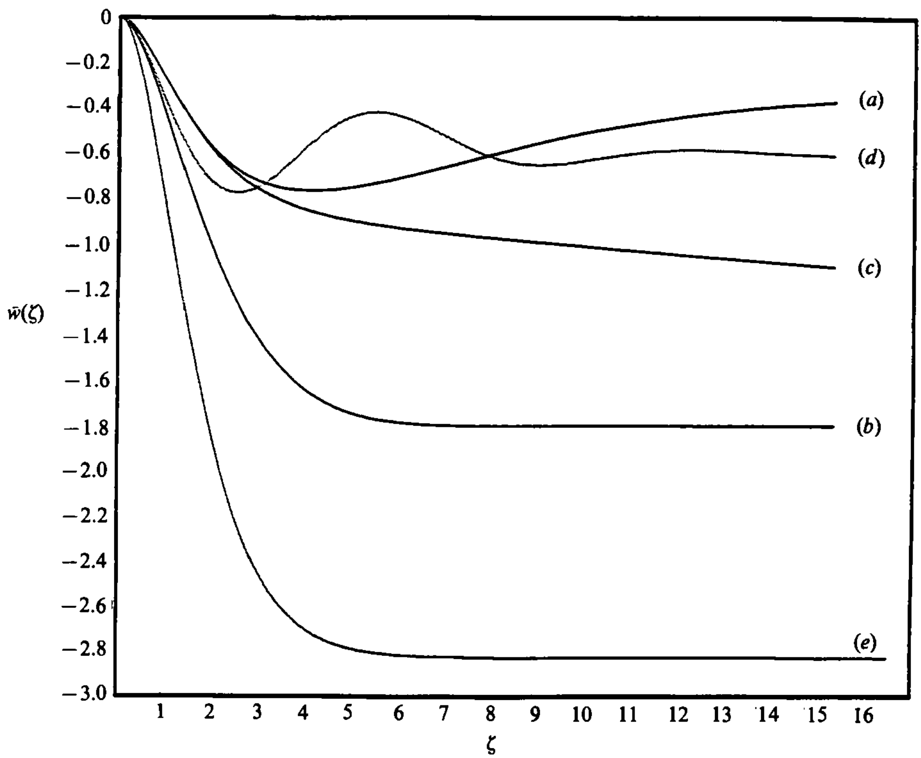

FIGURE 3. The function $\bar{w}(\zeta)$ for different values of γ, w_∞. (See figure 1 for values.)

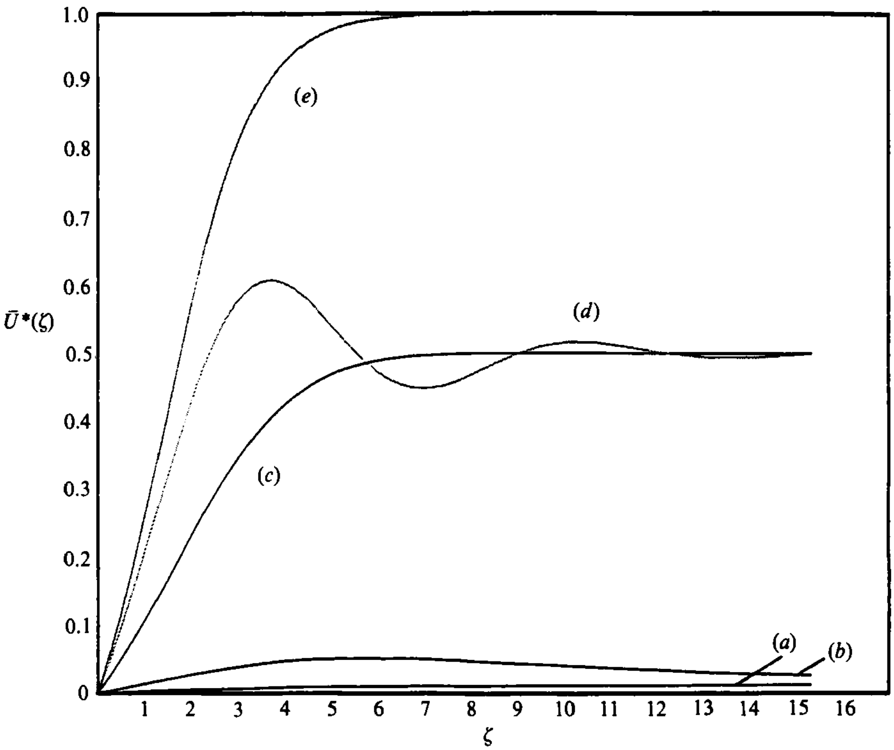

FIGURE 4. The function $\bar{U}^*(\zeta)$ for different values of γ, w_∞. (See figure 1 for values.)

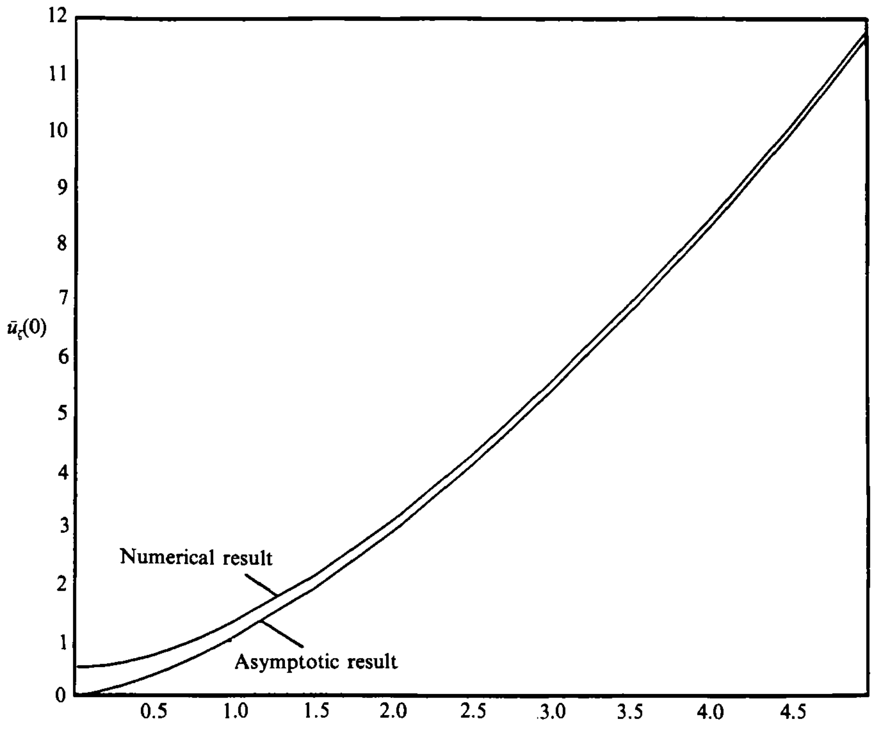

FIGURE 5. The shear $\bar{u}_\zeta(0)$ for $w_\infty^2 = 8\gamma$.

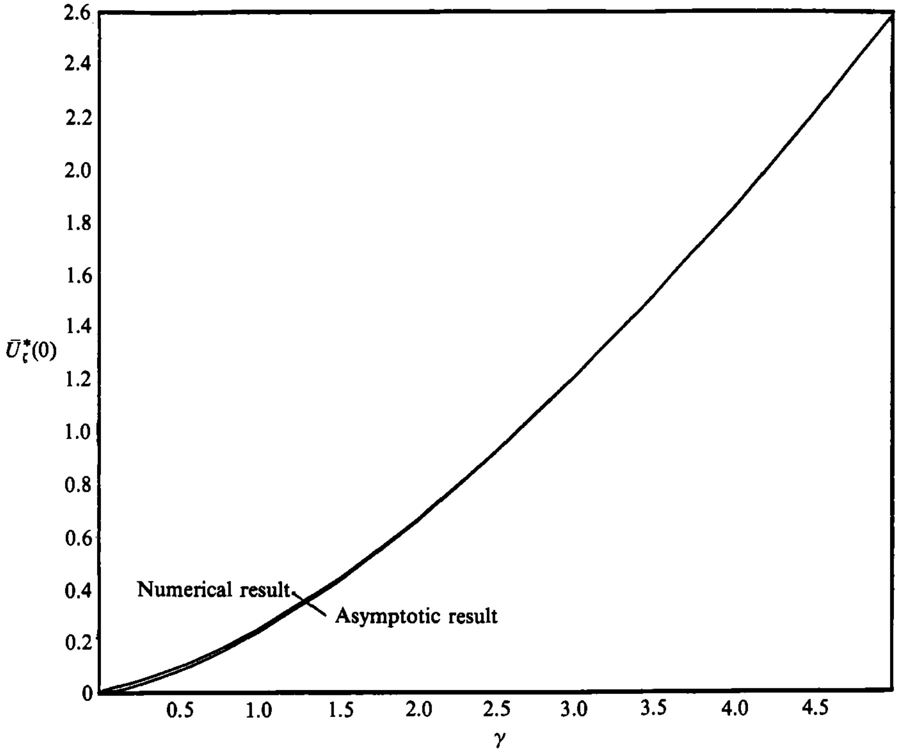

FIGURE 6. The shear $\bar{U}^*_\zeta(0)$ for $w_\infty^2 = 8\gamma$.

functions of η. Our limited calculations suggest that, for a given γ, the profiles become more oscillatory as w_∞ increases. A referee of this paper has suggested that the oscillatory behaviour of some of the solutions might be relevant to the question of which of the possible solutions are stable.

Further solutions of (2.9) were obtained for the case $w_\infty^2 = 8\gamma$ and our results are shown in figures 5 and 6. In figures 1–4 we have plotted the solution for $\gamma = 1$, $w_\infty = 8$ for comparison with the results for $w_\infty^2 \neq 8\gamma$. We recall that Davey obtained solutions of the large γ problem in a different context and that his results were obtained by neglecting the slowest decaying exponential solution at large ζ. Having made that approximation Davey found numerically that $w_\infty^2 = 8\gamma$ to the numerical accuracy of his calculations. Thus we expect that our results in figures 5 and 6 should reduce to those of Davey at sufficiently large values of γ. In fact, we have in these figures shown Davey's results expressed in our notation and we see that our results approach (3.7) for large γ. However, we stress at this point that there is no reason why the solutions of (2.9) obtained by rejecting a particular decaying exponential solution of that system at large ζ should be preferred, we hope to shed some light on the selection mechanism for the different solutions later in this paper.

4. Unsteady solutions of the interaction equations in the absence of a stagnation flow at infinity

We shall now discuss the solutions we have obtained for the unsteady form of the interaction equations. We restrict our attention to the case when the stagnation-point flow vanishes at infinity. This means that we are in effect discussing the finite-amplitude instability of Kármán's solution. Note, however, as kindly pointed out to the authors by a referee, the energy of the imposed disturbance is infinite. This means that our work is perhaps only relevant to a finite part of any physically realizable flow. In the Appendix we shall give a limited discussion of the more general problem when the stagnation flow does not vanish at infinity. In the absence of a stagnation flow at infinity we apply the conditions $U = 0, \zeta = 0, U \to 0, \zeta \to \infty$. The appropriately modified form of (2.10) is found to be

$$U_{\zeta\zeta} - U_T = 2\bar{u}U + \bar{w}U_\zeta, \tag{4.1a}$$

$$\bar{u}_{\zeta\zeta} - \bar{u}_T = \bar{u}^2 + |U|^2 - \bar{v}^2 + \bar{w}\bar{u}_\zeta, \tag{4.1b}$$

$$\bar{v}_{\zeta\zeta} - \bar{v}_T = 2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta, \tag{4.1c}$$

$$\bar{u} + \bar{w}_\zeta = 0, \tag{4.1d}$$

$$\bar{u} = \bar{w} = U = 0, \quad \bar{v} = 1, \quad \zeta = 0, \tag{4.1e}$$

$$\bar{u}, \bar{v}, U \to 0, \quad \zeta \to \infty, \tag{4.1f}$$

$$\bar{u} = \hat{u}(\zeta), \quad v = \hat{v}(\zeta), \quad U = \hat{U}(\zeta), \quad T = 0. \tag{4.1g}$$

The above system is parabolic in T and can be solved by marching forward in time from $T = 0$; we note here in passing that $\bar{w}$ cannot be specified arbitrarily, at $T = 0$ it must be deduced from $\hat{u}$ using the continuity equation. For large values of T the solution of (4.1) will approach Kármán's solution if that flow is stable. We can therefore regard (4.1) as the nonlinear initial-value instability problem for Kármán's rotating-disc flow. However, we should bear in mind that (4.1) describes only finite-amplitude disturbances with azimuthal wavenumber ± 2.

In the first instance, we restrict our attention to small initial perturbations from Kármán's solution, we therefore write

$$(\hat{u}, \hat{v}, \hat{U}) = (\bar{\bar{u}}, \bar{\bar{v}}, 0) + (u^*, v^*, \tilde{U}^*),$$

where $(\bar{\bar{u}}, \bar{\bar{v}})$ Kármán's solution and u^* etc. are small. We now substitute

$$(\bar{u}, \bar{v}, \bar{w}, U) = (\bar{\bar{u}} + \tilde{u}, \bar{\bar{v}} + \tilde{v}, \bar{\bar{w}} + \tilde{w}, \tilde{U})$$

in (4.1) and linearize to obtain the following decoupled problems:

$$\tilde{U}_{\zeta\zeta} - \tilde{U}_T = 2\bar{\bar{u}}\tilde{U} + \bar{\bar{w}}\tilde{U}_\zeta, \tag{4.2a}$$

$$\tilde{U} = 0, \quad \zeta = 0, \quad \infty, \tag{4.2b}$$

$$\tilde{U} = \tilde{U}^*, \quad T = 0, \tag{4.2c}$$

and

$$\tilde{u}_{\zeta\zeta} - \tilde{u}_T = 2\bar{\bar{u}}\tilde{u} - 2\bar{\bar{v}}\tilde{v} + \bar{\bar{w}}\tilde{u}_\zeta + \tilde{w}\bar{\bar{u}}_\zeta, \tag{4.3a}$$

$$\tilde{v}_{\zeta\zeta} - \tilde{v}_T = 2\bar{\bar{u}}\tilde{v} + 2\bar{\bar{v}}\tilde{u} + \bar{\bar{w}}\tilde{v}_\zeta + \tilde{w}\bar{\bar{v}}_\zeta, \tag{4.3b}$$

$$2\tilde{u} + \tilde{w}_\zeta = 0, \tag{4.3c}$$

$$\tilde{u} = \tilde{v} = \tilde{w} = 0, \quad \zeta = 0, \tag{4.3d}$$

$$\tilde{u}, \tilde{v} \to 0, \quad \zeta \to \infty, \tag{4.3e}$$

$$(\tilde{u}, \tilde{v}) = (u^*, v^*), \quad T = 0. \tag{4.3f}$$

It should be pointed out that in the above equations we have in effect assumed that $O(\tilde{U}^*) \sim O(u^*) \sim O(v^*)$; a modified form of the equations can be derived when

$O(\tilde{U}^*) \sim O(u^*)^{\frac{1}{2}}$. In that case a nonlinear term $|\tilde{U}|^2$ must be inserted into the right-hand side of the $\tilde{u}$ equation. This particular case would be important only if the $\tilde{U}$ problem were unstable. We can integrate (4.2), (4.3) formally by taking a Laplace transform in time. When the transform is inverted the nature of the solution will depend crucially on whether either of the eigenvalue problems

$$y'' - \sigma y = 2\bar{\bar{u}}y + \bar{\bar{w}}y', \tag{4.4a}$$

$$y(0) = y(\infty) = 0, \tag{4.4b}$$

or

$$y'' - \sigma y = 2\bar{\bar{u}}y - 2\bar{\bar{v}}z + \bar{\bar{w}}y' + x\bar{\bar{u}}', \tag{4.5a}$$

$$z'' - \sigma z = 2\bar{\bar{u}}z + 2y\bar{\bar{v}} + \bar{\bar{w}}z' + x\bar{\bar{v}}', \tag{4.5b}$$

$$2y + x' = 0, \tag{4.5c}$$

$$x(0) = y(0) = z(0) = y(\infty) = z(\infty) = 0, \tag{4.5d}$$

has an eigenvalue σ with positive real part. We shall now discuss these eigenvalue problems.

The above eigenvalue problems were solved numerically; no unstable eigenvalues of either system were found so we conclude that Kármán's solution is stable to small-amplitude perturbations of the type discussed here. In fact, no discrete stable eigenvalues were obtained either. This is because both eigenvalue problems have a continuous spectrum over part of the plane $\sigma_r < 0$. The origin of this continuous spectrum can be seen from (4.4a) by taking $\zeta \gg 1$. We then see that the two exponential solutions of the equation for y both decay if σ is within the parabola $\sigma_r = -\sigma_i^2/w_\infty^2$ where $w_\infty = \bar{\bar{w}}(\infty)$. Thus in this region we can always find a solution of (4.4) by combining the two independent solutions for y at infinity to satisfy the required condition at the wall. A similar continuous spectrum can be seen to exist for the system (4.5); we expect the continuous spectra to play an important role in the initial-value problems (4.2), (4.3). Indeed, since there is apparently no discrete spectrum associated with (4.4), (4.5) it is clear that the initial-value problem must in some sense be described completely by the continuous spectrum.

The initial-value problem can be solved by taking Laplace transforms and inverting for particular forms of the initial perturbation. These inversions cannot in general be carried out analytically but their large-time behaviour can be approximated asymptotically in a routine manner. Rather than use the Laplace transform method we shall instead look directly for the large-time behaviour of (4.2), (4.3). We restrict our attention to (4.2), but a similar approach can be used for (4.3).

Suppose that w_∞ denotes the limiting value of $\bar{\bar{w}}$ at large values of ζ, we choose to express $\tilde{U}$ in the form

$$\tilde{U} = M(\zeta, T)\exp\left[\tfrac{1}{2}(w_\infty\zeta) - \tfrac{1}{4}(w_\infty^2 T)\right]. \tag{4.6}$$

Here the function M satisfies the modified equation

$$M_{\zeta\zeta} + (w_\infty - \bar{\bar{w}})M_\zeta + \tfrac{1}{2}(-\bar{\bar{w}}w_\infty + w_\infty^2)M = M_T + 2\bar{\bar{u}}M. \tag{4.7}$$

At this stage we assume that all the exponential time dependence of the disturbance has been taken out by the substitution (4.6) so that M has only an algebraic dependence on T. Note, however, that the following discussion confirms that we can indeed obtain a consistent asymptotic solution of the problem by including all the exponential time dependence of the disturbance in the exponential term in (4.6). It is well known for the heat equation that the similarity variable $\zeta/T^{\frac{1}{2}}$ essentially replaces ζ when the initial-value problem is solved. Here the situation is slightly more complicated and, for large T, we must seek a solution of (4.7) for $\zeta = O(1)$ and $\zeta = O(T^{\frac{1}{2}})$. Thus for $\zeta = O(1)$ we write

$$M = T^j M_0(\zeta) + \dots,$$

where j is to be determined and M_0 satisfies the ordinary differential equation

$$M_0'' + (w_\infty - \bar{\bar{w}})M_0' + \tfrac{1}{2}(-\bar{\bar{w}}w_\infty + w_\infty^2)M_0 = 2\bar{\bar{u}}M_0. \tag{4.8}$$

This equation must be solved subject to $M_0 = 0$, $\zeta = 0$ and the resulting solution will then have $M_0 \sim c\zeta$ for large ζ, here c is an arbitrary constant which can be set equal to unity but whose actual value depends on the form of the initial disturbance. Now let us find the required form for M in the upper region, before doing so we note that the constant j is left unknown at this stage since it plays no role in the zeroth-order solution in the lower region. In the upper layer we write

$$M = T^{j+\frac{1}{2}}\tilde{M}_0(\tilde{\chi}) + \dots, \tag{4.9}$$

where $\tilde{\chi}$ is the similarity variable $\zeta/T^{\frac{1}{2}}$. Note here that the matching condition with the lower layer now requires that for small $\tilde{\chi}$, $\tilde{M}_0 \sim \tilde{\chi}$.

The equation to determine $\tilde{M}_0$ is found to be

$$\tilde{M}_0'' + \tfrac{1}{2}\tilde{\chi}M_0' - (j+\tfrac{1}{2})\tilde{M}_0 = 0. \tag{4.10}$$

Here a prime denotes a derivative with respect to $\tilde{\chi}$. In order that the disturbance decays to zero at large values of $\tilde{\chi}$ we must insist that $\tilde{M}_0$ behaves like the exponentially decaying solution of the above equation; since it must also go to zero like $\tilde{\chi}$, for small $\tilde{\chi}$ we can show that the required solution is

$$\tilde{M}_0 = \frac{2^{\frac{1}{2}}\exp(\frac{1}{8}\tilde{\chi}^2)\,\bar{U}^*(-n-\frac{1}{2}, 2^{-\frac{1}{2}}\tilde{\chi})}{U'(-n-\frac{1}{2}, 0)}. \tag{4.11}$$

Here n is an odd integer, $U(a, x)$ is a parabolic cylinder function whilst the constant j has been chosen to satisfy the matching condition at $\tilde{\chi} = 0$, this gives

$$j = -\tfrac{3}{2}, -\tfrac{5}{2}, -\tfrac{7}{2} + \dots. \tag{4.12}$$

It follows that the solution will be dominated by the $j = -\frac{3}{2}$ eigensolution for large enough values of the time, the overall amplitude of this and the other decaying modes can only be determined by solving the initial-value problem. In order to verify the above large-time behaviour of the solution of the linearized perturbation equation for $\tilde{U}$ we integrated (4.2) forward in time from $T = 0$ for the three cases:

case a $\qquad \tilde{U}^* = \zeta\exp(-\zeta^2),$

case b $\qquad \tilde{U} = \zeta\cos\zeta\exp(-\zeta),$

case c $\qquad \tilde{U} = \zeta\exp[-(\zeta-2)^2].$

We note that it is sufficient for us to consider real initial data for $\tilde{U}$ if we are solving (4.2). The results we obtained are shown in figure 7; in order to pick out the dominant exponential decay factor of $\tilde{U}$ we have plotted $(\log(T^{\frac{3}{2}}\tilde{U}_\zeta(0, T))_T$. On the basis of our discussion above we see that this quantity should tend to $-\frac{1}{4}w_\infty^2 \sim -0.2$ for large T. We see that each of the above cases leads to results consistent with our predictions. In fact it is easy to show that the correction to the growth rate in the limit of large T is $O(T^{-2})$, the results shown in figure 7 confirm this prediction. In order to see why

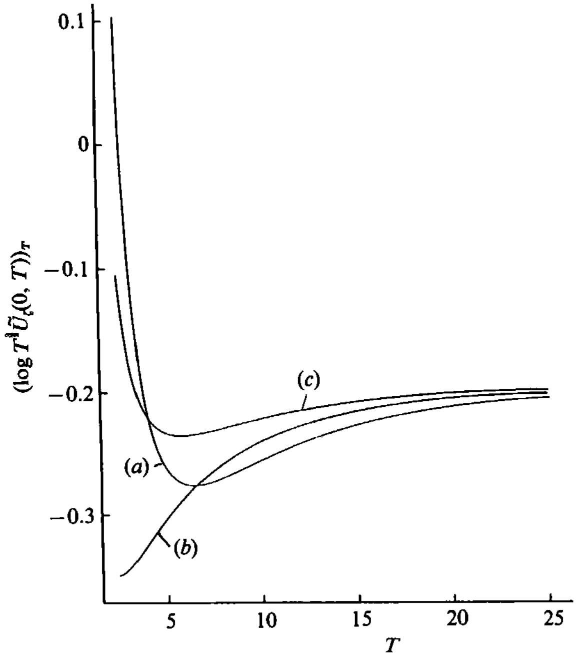

FIGURE 7. The growth rate $(\log T^{\frac{3}{4}}\tilde{U}_\zeta(0,T))_T$ for (a) $\tilde{U}^* = \zeta\exp(-\zeta^2)$; (b) $\tilde{U}^* = \zeta\cos\zeta\exp(-\zeta)$; (c) $\tilde{U}^* = \zeta\exp[-(\zeta-2)^2]$. The predicted growth rate at large T is $-\frac{1}{4}w_\infty^2 \approx -0.2$.

this is the case, we note, for example, that the difference between the calculated growth rates and their asymptotic value decreases by a factor of about 4 when T increases from 10 to 20. A similar analysis to that carried out above for (4.2) can be given for (4.3), again the outcome is that a two-layer structure is required to describe the large-time behaviour of the disturbance, furthermore the functions $\tilde{u}$, $\tilde{v}$ are found to decay exponentially for large T with the same decay rate as that found above. We note also that large-time instability analysis given above is related to that given by Bodonyi & Ng (1984) for swirling flows above discs; the authors thank a referee for pointing out that reference.

We shall now report on some calculations carried out for the full nonlinear problem (4.1) with initial conditions

$$\hat{u} = \hat{v} = 0, \quad \hat{U} = \delta\zeta\exp[-(\zeta-2)^2], \tag{4.13}$$

for $\delta = 0.35$, 0.45, 0.55, 0.65. The results obtained for the initial conditions given above are typical of those we have found for a wide range of disturbances. The results we found are shown in figure 8 where we have shown the growth rates

$$\left(\log\int_0^\infty \bar{u}^2\,d\zeta\right)_T, \quad \left(\log\int_0^\infty U^2\,d\zeta\right)_T, \quad \left(\log\int_0^\infty \bar{v}^2\,d\zeta\right)_T, \quad (\log(U_\zeta(0,T)\,T^{\frac{3}{4}})_T.$$

For the two smaller values of the amplitude constant δ we see that the disturbance decays to zero so that Kármán's solution is stable, note that figure 8(*d*) for $\delta = 0.35$ confirms the linear decay rate, -0.2, at sufficiently large values of T. The calculations for the two larger values of δ demonstrate that Kármán's solution is subcritically unstable. At a finite value of T our calculations encountered a singularity and could not be continued further. We did, of course, check that the

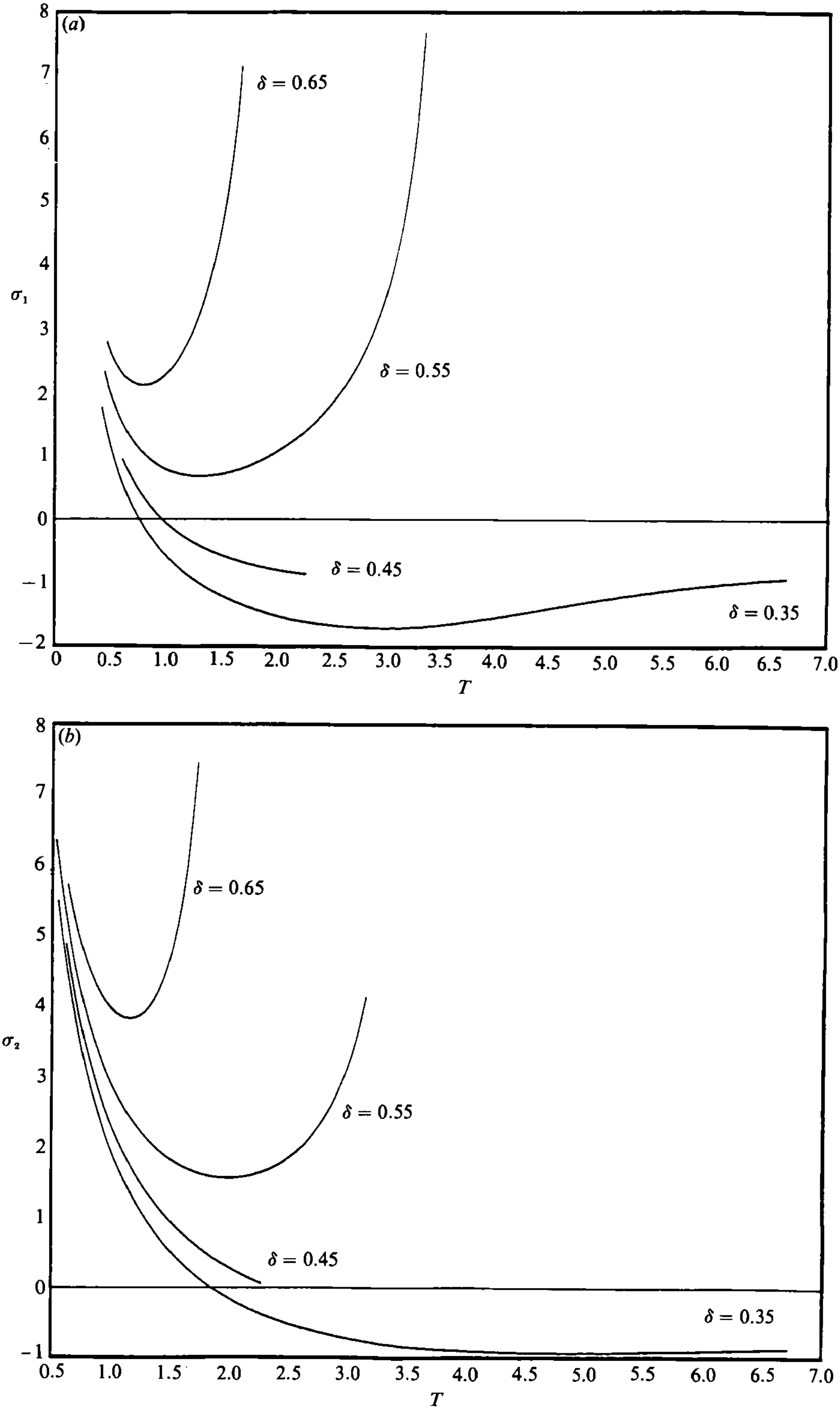

FIGURE 8 (a,b). For caption see facing page.

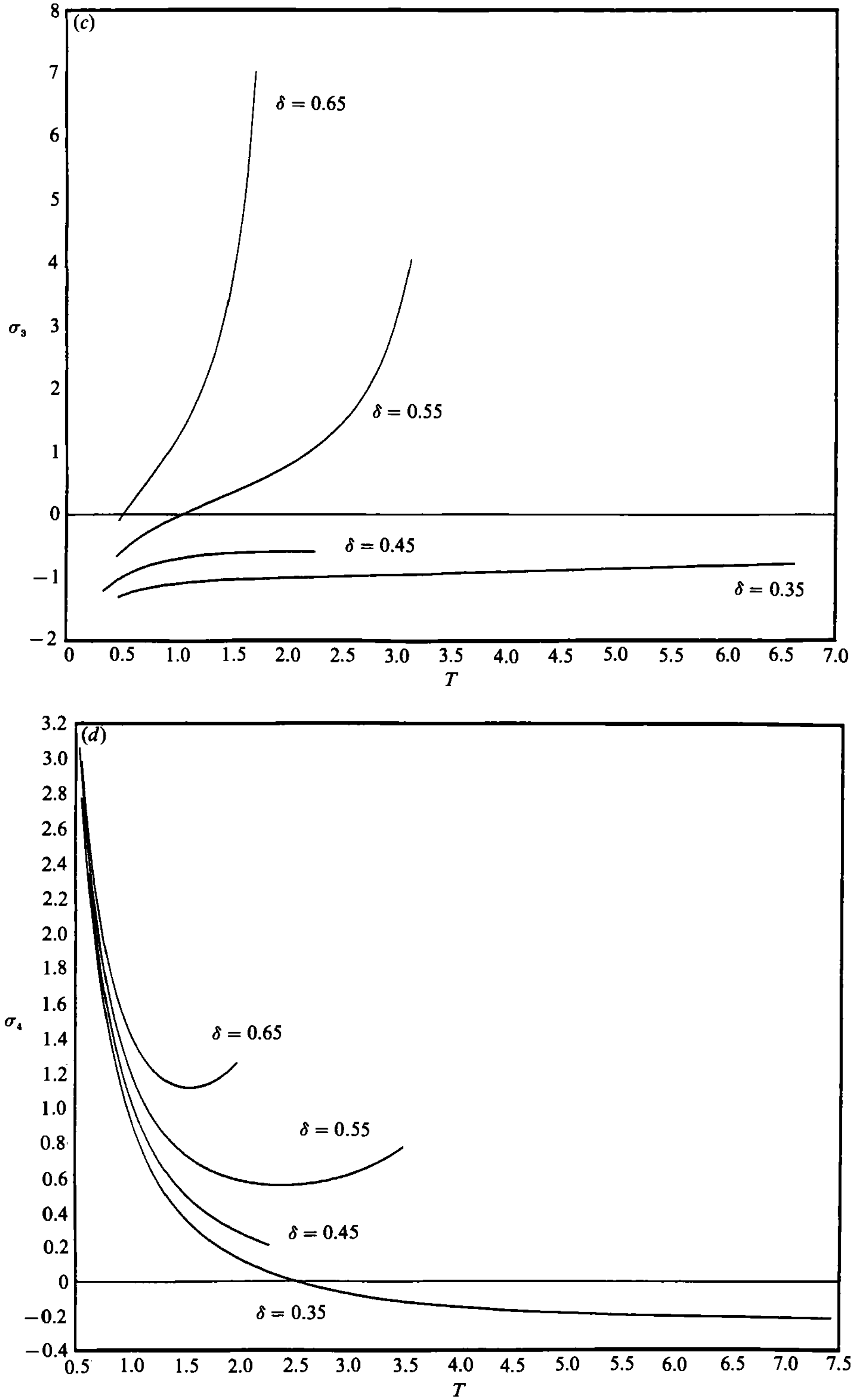

FIGURE 8. (*a*) The growth rate $\sigma_1 = (\log \int_0^\infty \bar{u}^2\, d\zeta)_T$ for $\delta = 0.35, 0.45, 0.55, 0.65$. (*b*) The growth rate $\sigma_2 = (\log \int_0^\infty \bar{v}^2\, d\zeta)_T$ for $\delta = 0.35, 0.45, 0.55, 0.65$. (*c*) The growth rate $\sigma_3 = (\log \int_0^\infty U^2\, d\zeta)_T$ for $\delta = 0.35, 0.45, 0.55, 0.65$. (*d*) The growth rate $\sigma_4 = (\log T^{\frac{2}{3}} U'(0,T))_T$ for $\delta = 0.35, 0.45, 0.55, 0.65$.

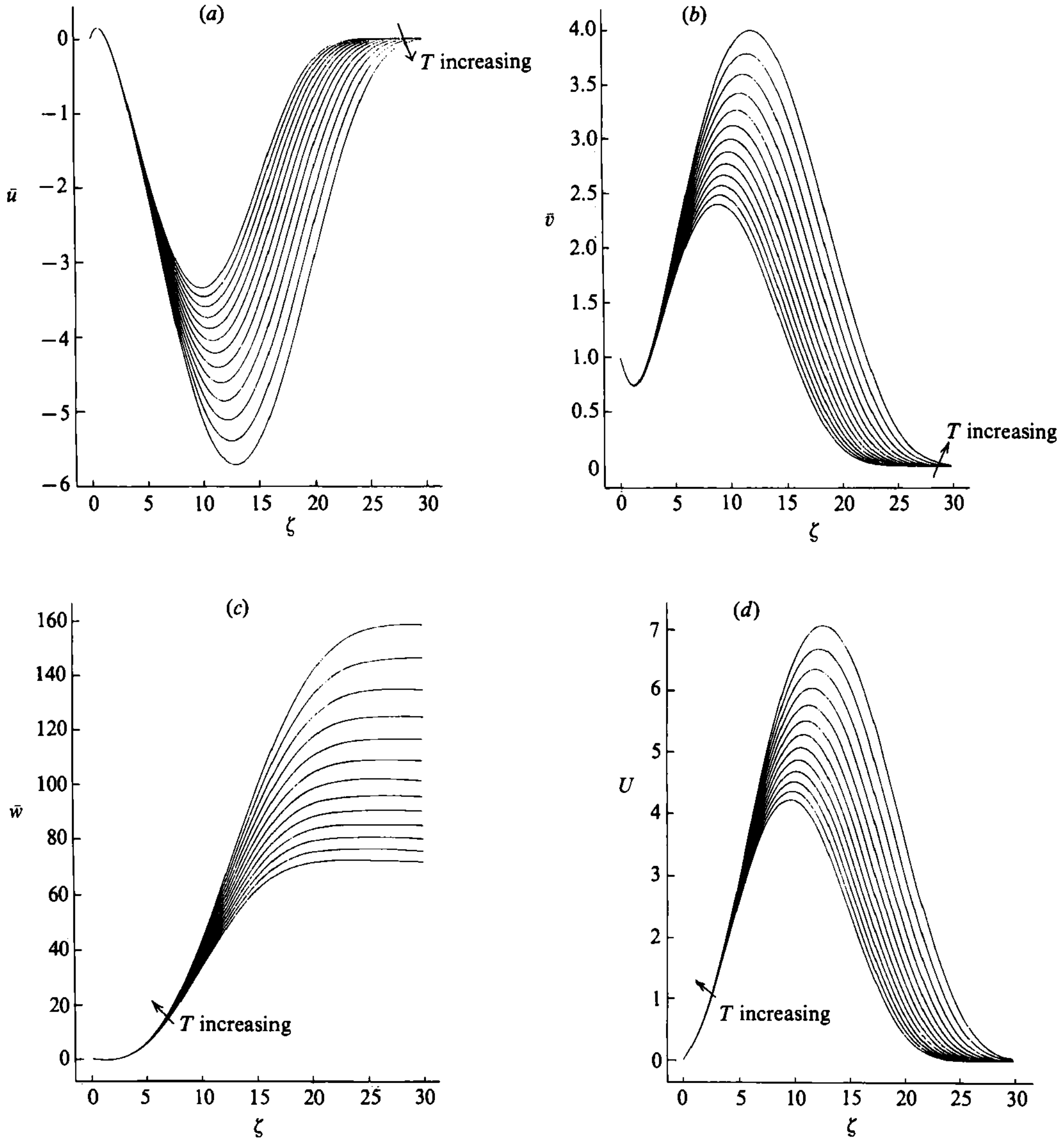

FIGURE 9. (a) The velocity field $\bar{u}(\zeta)$ for $\delta = 0.55$, $T = 3.505, 3.51, \ldots$. (b) The velocity field $\bar{v}(\zeta)$ for $\delta = 0.55$, $T = 3.505, 3.51, \ldots$. (c) The velocity field $\bar{w}(\zeta)$ for $\delta = 0.55$, $T = 3.505, 3.51, \ldots$. (d) The velocity field $U(\zeta)$ for $\delta = 0.55$, $T = 3.505, 3.51, \ldots$.

singularity remains when the ζ and T step lengths were decreased. In figure 9 we show the velocity field for the case $\delta = 0.55$, $T = 3.505$, 3.51, ... We see that as the singularity develops the velocity field spreads away from the wall. We note that as the singularity develops, the profiles for $\bar{u}, \bar{v}, U$ are of the same shape. Furthermore calculations for other initial disturbances confirmed the threshold type of response described above.

Thus, we have found that Kármán's rotating-disc solution is unstable to finite-amplitude infinite energy $n = \pm 2$ modes whereas in the linear regime we have stability to this type of disturbance. We have made no attempt to investigate the dependence of the required threshold amplitude of the instability on the form of the initial disturbance. For the case discussed above instability occurs when the disturbance velocity field is of size comparable to the unperturbed state. Thus, this particular type of disturbance is unlikely to be present in an experimental investigation so that it is unlikely that this disturbance would cause transition. However, it is not unreasonable to expect that a detailed investigation of a more general class of initial conditions would isolate more dangerous disturbances which might cause instability in an experimental facility with moderate background disturbances.

5. Singular solutions of the interaction equations

The calculations described in the previous section suggest that a singularity of the interaction equations (4.1) develops at a finite time. If we assume that as the singularity develops, inviscid effects dominate over most of the flow, then $\bar{u}_T \sim \bar{u}^2$ and it follows that we must have $\bar{u} \sim (\bar{T}-T)^{-1}$ as the singularity develops; a similar argument shows that $\bar{v}$ has this same scale. It then follows from the continuity equation and the momentum equations that if the thickness of the layer in which the disturbance develops is $O(\bar{T}-T)^{-\psi}$ then $\bar{w} \sim (\bar{T}-T)^{-(\psi+1)}$ with $\psi > 0$. However, an investigation of the matching problems associated with the outer edge of this layer suggests to the authors that only the case $\psi = \frac{1}{2}$ is possible and we therefore concentrate on that case.

We now define $\hat{\zeta}$ by $\hat{\zeta} = \zeta(\bar{T}-T)^{\frac{1}{2}}$ and then write

$$\bar{u} = \frac{\hat{u}_s(\hat{\zeta})}{\bar{T}-T} + \dots, \tag{5.1a}$$

$$\bar{v} = \frac{\hat{\psi}_s(\hat{\zeta})}{\bar{T}-T} + \dots, \tag{5.1b}$$

$$\bar{w} = \frac{\hat{w}_s(\hat{\zeta})}{(\bar{T}-T)^{\frac{3}{2}}} + \dots, \tag{5.1c}$$

$$U = \frac{\hat{U}_s(\hat{\zeta})}{\bar{T}-T} + \dots. \tag{5.1d}$$

The zeroth-order problem for $\hat{U}_s, \hat{u}_s, \hat{v}_s, \hat{w}_s$ is obtained by substituting the above expansions into the interaction equations and equating dominant terms in the limit of small $\bar{T}-T$. We find that $\hat{U}_s, \hat{v}_s$ satisfy the same equations so, if $\hat{v}_s \neq 0$, we can write

$$\hat{U}_s = (1+\bar{\delta}\hat{v}_s)^{\frac{1}{2}},$$

with $\bar{\delta} > -1$. If we then eliminate $\hat{u}_s$ from the radial momentum equations using continuity we arrive at the following coupled pair of equations for $\hat{v}_s$, $\hat{w}_s$:

$$(2\hat{w}_s - \hat{\zeta})\,\hat{w}_{s\hat{\zeta}\hat{\zeta}} + 2\hat{w}_{s\hat{\zeta}} = 4\bar{\delta}\hat{v}_s^2 + \hat{w}_{s\hat{\zeta}}^2, \tag{5.2a}$$

$$(2\hat{w}_s - \hat{\zeta})\,\hat{v}_{s\hat{\zeta}} - 2\hat{v}_s\,\hat{w}_{s\hat{\zeta}} = -2\hat{v}_s, \tag{5.2b}$$

and since viscous effects are negligible in the derivation of (5.2) we must solve these equations subject to

$$\hat{w}_s = 0, \quad \hat{\zeta} = 0. \tag{5.3}$$

For positive values of $\bar{\delta}$, an exact solution of (5.2) is

$$\hat{w}_s = \hat{\zeta}, \quad 4\bar{\delta}\hat{v}_s^2 = 1, \tag{5.4a}$$

and if $\bar{\delta} = 0$ an exact solution is

$$\hat{w}_s = 2\hat{\zeta}. \tag{5.4b}$$

We note here that if $\bar{\delta} = 0$ there is no coupling between the $\hat{v}_s, \hat{w}_s$ equations so we can take $\hat{v}_s = 0$. However, neither of the above exact solutions is bounded at infinity, so they are not acceptable solutions. In fact it can be shown that (5.2) does not admit solutions which have $\hat{v}_s, \hat{w}_{s\zeta}$ tending to zero at infinity. We shall see below that the required exponential decay of the singular solutions is taken care of by viscous effects.

We now choose some positive constant $\bar{\zeta} > 0$ and restrict our attention to the solution of (5.2) on $[0, \bar{\zeta}]$. In order to solve (5.2) it is useful to note that, for a given $\bar{\zeta}$, these equations are invariant under the transformation:

$$\tilde{\zeta} = 2\frac{\hat{\zeta}}{\bar{\zeta}} - 1, \quad \tilde{w}_s = 2\frac{\hat{w}_s}{\bar{\zeta}} - \tfrac{1}{2}, \quad \tilde{v}_s = \hat{v}_s, \tag{5.5}$$

so that $\tilde{v}_s, \tilde{w}_s$ now satisfy (5.2) with $\hat{\zeta}$ replaced by $\tilde{\zeta}$ and subject to

$$\tilde{w}_s = -\tfrac{1}{2}, \quad \tilde{\zeta} = -1. \tag{5.6}$$

The form of (5.2) enables us to seek solutions which have $\tilde{v}_s, \tilde{w}_s$ respectively even and odd functions of $\tilde{\zeta}$ on $[-1, 1]$. We can show that the small $\tilde{\zeta}$ solution of (5.2) having the required symmetry is

$$\tilde{v}_s = \frac{1}{2\bar{\delta}^{\frac{1}{2}}}\tilde{w}_{s\tilde{\zeta}}, \quad \tilde{w} = \tilde{\zeta} + \beta\tilde{\zeta}^3 + \dots, \tag{5.7a}$$

for $\bar{\delta} \neq 0$ whilst for $\bar{\delta} = 0$ we have

$$\tilde{w}_s = 2\tilde{\zeta} + \beta\tilde{\zeta}^{\frac{5}{3}} + \dots, \tag{5.7b}$$

In (5.7a, b) the constant β is unknown at this stage; we note again that in the special case $\bar{\delta} = 0$ it is sufficient to set $\tilde{v}_s = 0$ since there is no coupling between the radial and azimuthal momentum equations. It should also be noted that both of the above small $\tilde{\zeta}$ solutions for $\tilde{w}_s$ are above the line $2\tilde{w}_s = \tilde{\zeta}$ on which (5.2) is singular. The constant β is now chosen such that the small $\tilde{\zeta}$ solutions lead to functions $\tilde{w}_s$ satisfying $2\tilde{w}_s = 1$, $\tilde{\zeta} = 1$. Since $\tilde{w}_s$ is an odd function of $\tilde{\zeta}$ it follows that the inviscid boundary (5.6) condition at the wall, is therefore satisfied. Thus we choose β such that the solutions of (5.2) corresponding to (5.7) are singular at the points $\tilde{\zeta} = \pm 1$. In fact, for positive $\bar{\delta}$, it is easy to show that the only solution of (5.2) satisfying (5.6), (5.7a) can be written down in closed form and is

$$\tilde{v}_s = \frac{1}{2\bar{\delta}^{\frac{1}{2}}}\tilde{w}_{s\tilde{\zeta}}, \quad \tilde{w}_s = \tfrac{1}{2}\tilde{\zeta} + \frac{\sin(\pi\tilde{\zeta})}{2\pi}. \tag{5.8a, b}$$

We were unable to find an exact solution of (5.2) corresponding to (5.7b) but by numerical integration we found that the required value of β is $\beta = -2.129$. It has been pointed out to the authors by an anonymous referee that (5.8b) has in fact been found previously by Bodonyi & Stewartson (1977), and Banks & Zaturska (1979). The latter authors were concerned with singular solutions relevant to counter-rotating flow over discs and spheres respectively. The functions $\tilde{w}_s, \tilde{w}_{s\zeta}$ for $\bar{\delta} = 0$,

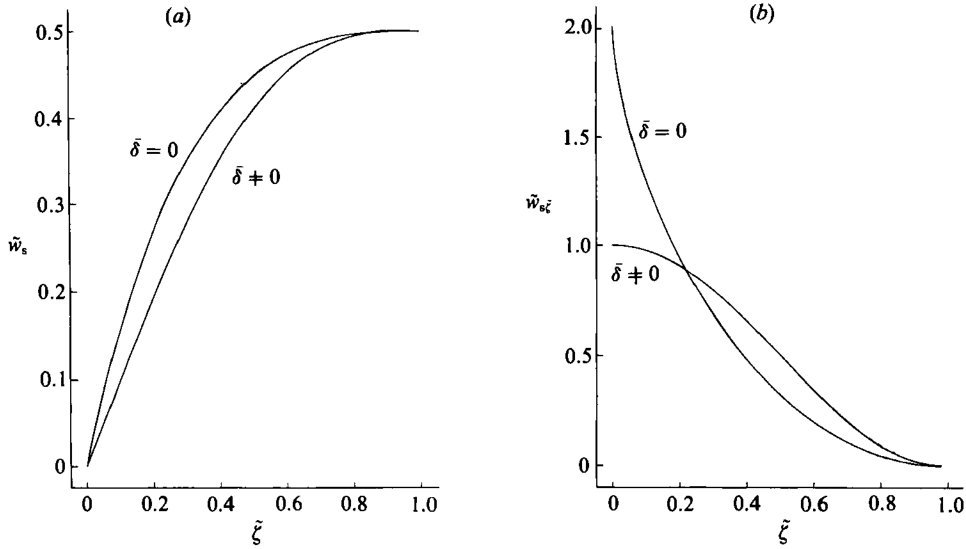

FIGURE 10. (*a*) The function $\tilde{w}_s$ for $\bar{\delta} = 0$, $\bar{\delta} \neq 0$. (*b*) The function $\tilde{w}_{s\tilde{\zeta}}$ for $\bar{\delta} = 0$, $\bar{\delta} \neq 0$.

$\bar{\delta} \neq 0$ are shown in figure 10. We must now show how the solutions determined above in the region $\zeta = O(\bar{T}-T)^{-\frac{1}{2}}$ connect with viscous structure at the boundaries and near $\hat{\zeta} = \bar{\zeta}$. We shall concentrate on the solution which connects with (5.8), a similar analysis can be given for the second solution.

We first note from (5.8*a*, *b*) that for $\tilde{\zeta} \sim \pm 1$

$$2\tilde{v}_s \bar{\delta}^{-\frac{1}{2}} = \tilde{w}_{s\tilde{\zeta}}, \quad \tilde{w}_s \sim \pm\tfrac{1}{2} \pm \tfrac{1}{12}\pi^2(\tilde{\zeta} \pm 1)^3 + \dots .$$

The required structure at the wall is found by noting that, in view of (5.5), the above small $\tilde{\zeta}$ solutions of (5.2) imply that $\bar{u}, \bar{v}, \bar{w}$, and U all become $O(1)$ when $\tilde{\zeta}+1$ becomes $O(\bar{T}-T)^{-\frac{1}{2}}$. This suggest that the inviscid structure found above connects with a viscous boundary layer of thickness $O(1)$ in terms of ζ at the wall. Thus the solution for $\zeta = O(1)$ is found by solving the full interaction equations (4.1) subject to the no-slip condition at the wall and with matching conditions at infinity implied by the small $\tilde{\zeta}$ limit of the inviscid solution. This boundary layer is clearly passive and can be calculated in a manner similar to that which we now outline for the more complicated viscous structure near $\hat{\zeta} = \bar{\zeta}$. It is easy to show from (5.8) that when $2\tilde{w}_s = 1$, i.e. near $\tilde{\zeta} = 1$, the functions $\tilde{v}_s, 2\tilde{w}_s - \tilde{\zeta}$ are respectively quadratic and cubic functions of $\tilde{\zeta} - \bar{\zeta}$. This implies that the limiting form of the inviscid solution near $\zeta = \bar{\zeta}/(\bar{T}-T)^{\frac{1}{2}}$ gives

$$\bar{u} = -\frac{\pi^2}{2\bar{\zeta}^2}\left(\zeta - \frac{\bar{\zeta}}{(\bar{T}-T)^{\frac{1}{2}}}\right)^2 + \dots , \tag{5.9a}$$

together with similar expansions for $\bar{u}, U$ whilst $\bar{w}$ takes the form

$$\bar{w} = \left(\frac{\bar{\zeta}}{(\bar{T}-T)^{\frac{1}{2}}}\right)_T + \frac{\pi^2}{3\bar{\zeta}^2}\left(\zeta - \frac{\bar{\zeta}}{(\bar{T}-T)^{\frac{1}{2}}}\right)^3 + \dots , \tag{5.9b}$$

and we note that the expansions for $\bar{v}, U$ are similar to (5.9*a*), The above expansions show that $\bar{u}, \bar{v}, U$ and $\bar{w} - [\bar{\zeta}/(\bar{T} = T)^{\frac{1}{2}}]_T$ become $O(1)$ within a layer of depth $O(1)$ in terms of ζ around the position $\zeta = \bar{\zeta}(\bar{T}-T)^{\frac{1}{2}}$. Hence, we therefore look for a solution of the full interaction equations in a layer of depth $O(1)$ propagating in the positive

ζ-direction with speed $[\bar{\zeta}/(\bar{T}-T)^{\frac{1}{2}}]_T$. The appropriate boundary-layer variable is therefore defined by

$$\zeta^* = \zeta - \frac{\bar{\zeta}}{(\bar{T}-T)^{\frac{1}{2}}},$$

and we seek a solution of (4.1) of the form

$$\bar{u} = u^*(T, \zeta^*) + O(\bar{T}-T)^{\frac{1}{2}}, \tag{5.10a}$$

for $\bar{u}$ and similar expansions for $\bar{v}, U$ whilst for $\bar{w}$ we write

$$\bar{w} - \left[\frac{\bar{\zeta}}{(\bar{T}-T)^{\frac{1}{2}}}\right]_T = w^*(T, \zeta^*) + O(\bar{T}-T)^{\frac{1}{2}}. \tag{5.10b}$$

We note that u^*, v^*, w^*, U^* all depend on T rather than $\bar{T}-T$. This means that the required decay of the singular solution at infinity is not determined in terms of a similarity variable involving $\bar{T}-T$. In order to see why this should be the case it is helpful to consider the model problem

$$\Psi_t + \frac{\Psi_x}{2(\bar{t}-t)^{\frac{1}{2}}} = \Psi_{xx}, \tag{5.11}$$

subject to $\quad \Psi = F(x), \quad t = 0, \quad \Psi \to 0, \quad |x| \to \infty. \quad (5.12)$

The above system is singular at $t = \bar{t}$. If the function F has compact support then the solution of the above initial-value problem is given by

$$\Psi = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} F(\tilde{x}) \exp\{-[x - \tilde{x} - 1/(\bar{t}-t)^{\frac{1}{2}}]^2\} \, d\tilde{x}. \tag{5.13}$$

Thus even though the differential equation for Ψ is singular at $t = \bar{t}$, when calculated in a coordinate frame moving to the right with speed $1/(\bar{t}-t)^{\frac{1}{2}}$, Ψ is not singular; however we note that this speed tends to infinity as $t \to \bar{t}_-$.

The structure discussed above for the model equation is precisely the type of behaviour implied by (5.10). If the expansions (5.10) together with the similar expansions for $\bar{v}, U$ are substituted into (4.1) and the leading-order terms are retained in the limit $T \to \bar{T}_-$ we find that u^*, v^*, w^*, U^* satisfy the full unsteady interaction equations but with ∂_ζ replaced by ∂_{ζ^*}. The appropriate boundary conditions are obtained by matching with the inviscid solution and insisting that, apart from w^*, all the velocity components should tend to zero at infinity. The matching conditions obtained from the inviscid solution yield:

$$u^* \sim -\frac{1}{2}\frac{\pi^2}{\bar{\zeta}^2}\zeta^{*2}, \quad \bar{\delta}^{\frac{1}{2}} v^* \sim \frac{1}{2}\frac{\pi^2}{\bar{\zeta}^2}\zeta^{*2}, \quad w^* \sim \frac{1}{3}\frac{\pi^2}{\bar{\zeta}^2}\zeta^{*3}, \quad \bar{\delta}^{\frac{1}{2}} U^* \sim \tfrac{1}{2}(1+\bar{\delta})^{\frac{1}{2}}\frac{\pi^2}{\bar{\zeta}^2}\zeta^{*2}, \quad \zeta^* \to -\infty, \tag{5.14}$$

whilst at infinity we require

$$u^*, v^*, U^*, \quad \to 0, \quad \zeta^* \to \infty. \tag{5.15}$$

We are, of course, interested in the solution of the partial differential system for u^*, v^*, w^*, U^* in the limit of small $\bar{T}-T$, hence we can obtain the required solution by finding a similarity solution using the variable $\chi = \zeta/\bar{T}^{\frac{1}{2}}$ and expanding the solution about $T = \bar{T}$. We therefore write

$$u^* = \frac{\breve{u}(\chi)}{T}, \quad v^* = \frac{\breve{v}(\chi)}{T}, \quad w^* = \frac{\breve{w}(\chi)}{(T)^{\frac{1}{2}}}, \quad U^* = \frac{\breve{U}(\chi)}{T}, \tag{5.16}$$

and we can then show that the ordinary differential system to determine $\breve{u}, \breve{v}, \breve{w}, \breve{U}$ is:

$$\breve{u}^2-\breve{v}^2+\breve{U}^2+\breve{w}\breve{u}_\chi = \breve{u}_{\chi\chi}+\tfrac{1}{2}\chi\breve{u}_\chi+\breve{u}, \tag{5.17a}$$

$$2\breve{u}\breve{v}+\breve{w}\breve{v}_\chi = \breve{v}_{\chi\chi}+\breve{v}+\tfrac{1}{2}\chi\breve{v}_\chi, \tag{5.17b}$$

$$2\breve{u}\breve{U}+\breve{w}\breve{U}_\chi = \breve{U}_{\chi\chi}+\breve{U}+\tfrac{1}{2}\chi\breve{U}_\chi, \tag{5.17c}$$

$$2\breve{u}+\breve{w}_\chi = 0, \tag{5.17d}$$

subject to

$$\breve{u}, \breve{v}, \breve{U}\to 0, \quad \chi\to\infty, \tag{5.17e}$$

and

$$\breve{u} \sim \frac{1}{2}\frac{\pi^2}{\bar{\zeta}^2}\chi^2, \quad \bar{\delta}^{-\frac{1}{2}}\breve{v} \sim \frac{1}{2}\frac{\pi^2}{\bar{\zeta}^2}\chi^2, \quad \breve{w} \sim \frac{1}{3}\frac{\pi^2}{\bar{\zeta}^2}\chi^3, \quad \bar{\delta}^{\frac{1}{2}}\breve{U} \sim (1+\bar{\delta})^{\frac{1}{2}}\frac{1}{2}\frac{\pi^2}{\bar{\zeta}^2}\chi^2, \quad \chi\to-\infty. \tag{5.17f}$$

Now let us discuss the nature of the flows associated with the singular solutions found above. We note that the singular solution consists of an inviscid region of depth $O(\bar{T}-T)^{-\frac{1}{2}}$ together with viscous boundary layers at the wall and at the edge of the inviscid region. The singular solution with $\delta = 0$ corresponds to an axisymmetric stagnation-point type of flow.

The solution (5.8) corresponds to a three-dimensional stagnation-point type of singular flow. In order to see why this is the case, we note that in the inviscid region the velocity field corresponding to (5.8) with respect to a Cartesian coordinate system is

$$\frac{2\boldsymbol{u}}{\Omega} = \left(\left[x(\Phi-1)-\frac{y}{\bar{\delta}^{\frac{1}{2}}}\right]\tilde{w}_s'(\bar{T}-T)^{-1}, \quad \left[\frac{x}{\bar{\delta}^{\frac{1}{2}}}-y(1+\Phi)\right]\tilde{w}_s'(\bar{T}-T)^{-1}, \quad 2\left(\frac{\nu}{\omega}\right)^{\frac{1}{2}}\tilde{w}_s(\bar{T}-T)^{-\frac{3}{2}}\right)+\ldots, \tag{5.18}$$

Here $\Phi = (1+\bar{\delta}^{-1})^{\frac{1}{2}}$ and we note that the above velocity field cannot be transformed to plane stagnation-point form by rotating the (x,y)-axes; however, a flow of the latter type can be found by noting that, in addition to (5.8), the interaction equations in the inviscid equation have the further exact solution

$$\tilde{v}_s = 0, \quad \tilde{U}_s = \tfrac{1}{2}\tilde{w}_{s\tilde{\zeta}}, \quad \tilde{w}_s = \tfrac{1}{2}\tilde{\zeta}+\frac{\sin(\pi\tilde{\zeta})}{2\pi}. \tag{5.19}$$

With respect to a Cartesian coordinate frame the above solution corresponds to

$$\frac{\boldsymbol{u}}{\Omega} = \left(0, \quad -y\tilde{w}_s', \quad \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\tilde{w}_s\right)+\ldots, \tag{5.20}$$

which is of plane stagnation-point form.

It remains now for us to determine if the singular solutions derived above describe the singularity found numerically in the previous section. We note here that in many other calculations we found singular velocity fields similar to those shown in (4.3). In fact, our calculations suggest that over most of the range of values of ζ the functions $\bar{u}, \bar{v}, U$ differ only by a scale factor. This is entirely consistent with the singularity structure (5.18) and in fact $\bar{w}$ in figure 9(*c*) is clearly as predicted by (5.8*b*). In addition the function $\bar{v}$ is as predicted by (5.8*a*) so it would seem that the second type of singularity is the one encountered numerically. The results shown in figure 9 are

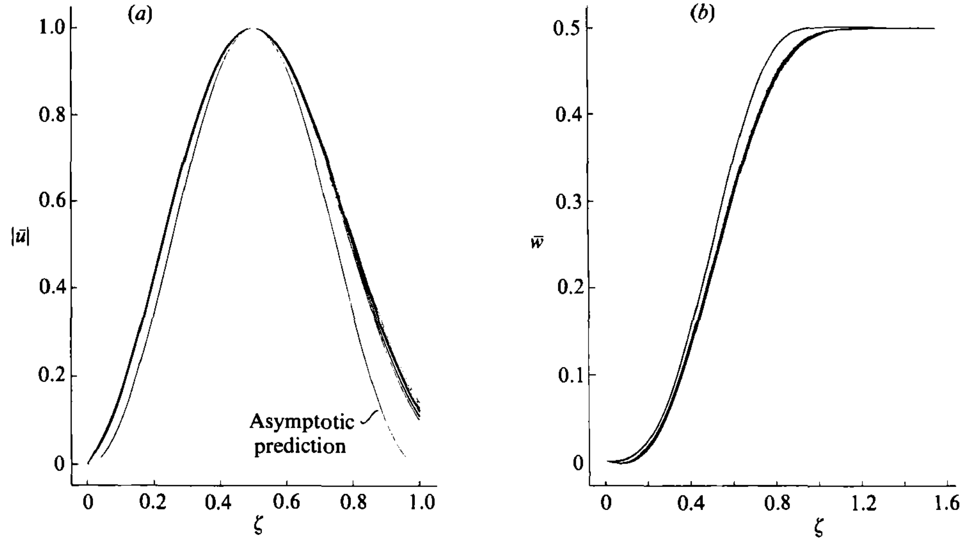

FIGURE 11. (*a*) A comparison between the collapsed data for $\bar{u}$ from figure 9(*a*) and the singular solution. (*b*) A comparison between the collapsed data for $\bar{w}$ from figure 9(*c*) and the singular solution.

typical of what we obtained in many other cases, in order to show conclusively that the singularity described above is the one found numerically we now collapse the data of figure 9 into a form which can easily be compared with (5.8).

In order to collapse the results in figure 9 we carried out the following procedure. First, we take the curves in figure 9(*a*) and rescale the horizontal and vertical scales such that the maximum of $\bar{u}$ always occur at $\zeta = \frac{1}{2}$ and has the value 1. We then rescale the horizontal scales for the corresponding curves of figure 9(*c*) and scale $\bar{w}$ such that $\bar{w} \rightarrow \frac{1}{2}$, $\zeta \rightarrow \infty$. Figure 11 shows the collapsed data for $\bar{u}$, $\bar{w}$ obtained in this way together with the inviscid solution (5.18). We see that the collapsed data is consistent with the predicted flow. We note that the data of figures 9(*b*) and 9(d) when collapsed in the same way agree equally well with predicted inviscid flow. Finally, we point out that in all our calculations the breakdown found was always of the type indicated in figure 11.

6. Conclusion

We have seen in the previous section that sufficiently large initial perturbations to Kármán's rotating-disc flow lead to the development of a singularity of the Navier–Stokes equations.

If we assume that the structure we have found can be induced experimentally then a question of some importance is that of how the singularity can be controlled through the Navier–Stokes equations once it has begun. Since no terms in the Navier–Stokes equations have been neglected in the analysis leading to the singularity it might appear that the singularity must be controlled by an alternative set of field equations. However, we do not believe that is the case, more precisely we believe that once the singularity has begun to develop the velocity profiles associated with it will be massively unstable to inviscid modes with aximuthal wavenumbers $\neq \pm 2$; these modes will then grow and prevent the further development of the singularity. The latter conjecture could, of course, be verified by Navier–Stokes simulations, we do not attempt such an investigation here. Certainly the modes discussed by Gregory *et al.* (1953) would be unstable in the initial stages of the development of the $n = \pm 2$ modes. In addition, time-dependent version of the latter 'crossflow' mode would also be unstable. It should be remembered that (2.3) can only be valid for finite range of values of the radial variable. Thus the expansion (2.3) is only valid in some local region of a more complicated flow structure; for example we could consider the situation when the disc is finite and some three-dimensional trailing-edge flow must be matched onto (2.3). It is possible that the singularity structure we have found could be destroyed by the flow away from the region where (2.3) is valid.

The typical size of initial amplitude associated with (4.13) required to cause the development of a singularity was found to be comparable with a typical basic state velocity. Thus, the breakdown we have described could only be provoked by a disturbance amplitude unlikely to be present in an experimental situation. However, it would be very surprising if the initial amplitude required to induce the singularity could not be significantly reduced by allowing a much more general initial perturbation. A possible way of isolating the most dangerous type of initial perturbation would be to formulate the energy stability problem associated with the interaction equations.

Another open question following our discussion in §3 is which, if any, of the different equilibrium states for non-zero γ are most physically relevant. Our linear instability analysis in the Appendix strongly suggests that all those equilibrium state with $w_\infty^2 < 8\gamma$ are not relevant physically since they are linearly unstable. It remains an open question as to whether nonlinear effects are able to further reduce the class of physically realizable flows.

P.H. wishes to thank SERC and AFOSR for partial support of this work. The authors would also like to thank the referees for useful comments on the original version of this paper. Also Professor J. T. Stuart has pointed out to us that he and Dr S. Allen had previously derived (4.1) in some unpublished work.

Appendix. The linear instability problem for the steady states with $\gamma \neq 0$

Here we shall discuss the instability of the steady equilibrium solutions of (2.9) with $N = 0$. We denote this steady state by $(\bar{\bar{u}}, \bar{\bar{v}}, \bar{\bar{w}}, \bar{U})$ and suppose that we consider the instability of this flow to a time-dependent small perturbation $(\tilde{u}, v\tilde{v}, \tilde{w}, \tilde{U})$ such that

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{U}) = (u^*, v^*, w^*, U^*), \quad T = 0. \tag{A 1}$$

From (2.10) the linearized perturbation equations for $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{U})$ are

$$\left.\begin{aligned}
&\tilde{U}_T + 2\bar{\bar{u}}\tilde{U} + 2\tilde{u}\bar{U} + \bar{\bar{w}}\tilde{U}_\zeta + \tilde{w}\bar{U}_\zeta = \tilde{U}_{\zeta\zeta},\\
&\tilde{u}_T + 2\bar{\bar{u}}\tilde{u} + 2\gamma^2\bar{U}\tilde{U} - 2\bar{\bar{v}}\tilde{v} + \bar{\bar{w}}\tilde{u}_\zeta + \tilde{w}\bar{\bar{u}}_\zeta = \tilde{u}_{\zeta\zeta},\\
&\tilde{v}_T + 2\bar{\bar{u}}\tilde{v} + 2\tilde{u}\bar{\bar{v}} + \bar{\bar{w}}\tilde{v}_\zeta + \tilde{w}\bar{\bar{v}}_\zeta = \tilde{v}_{\zeta\zeta},\\
&2\tilde{u} + \tilde{w}_\zeta = 0,
\end{aligned}\right\} \tag{A 2}$$

subject to

$$\left.\begin{aligned}
&\tilde{u} = \tilde{v} = \tilde{w} = \tilde{U} = 0, \quad \zeta = 0,\\
&\tilde{u} = \tilde{v} = \tilde{U} = 0, \quad \zeta = \infty.
\end{aligned}\right\} \tag{A 3}$$

and (A 1). Again the initial-value problem can be solved by a Laplace transform in T but the form of (A 2) means that the resulting ordinary differential equations in ζ

must be solved numerically. Following our approach in §4 we consider the eigenvalue problem $\sigma = \sigma(\gamma)$ obtained by replacing ∂_T by σ in (A 2) and applying (A 3). The structure of the disturbed flow at $\zeta = \infty$ is then found by letting $\zeta \to \infty$ in (A 2) after replacing ∂_T by σ and ∂_ζ by m; the appropriate equations to determine m are

$$(m^2 - w_\infty m - \sigma)\,\tilde{U} = 2\tilde{u}, \tag{A 4a}$$

$$(m^2 - w_\infty m - \sigma)\,\tilde{u} = 2\gamma\tilde{U}, \tag{A 4b}$$

$$(m^2 - w_\infty m - \sigma)\,\tilde{v} = 0. \tag{A 4c}$$

Thus m is given by

$$2m = w_\infty \pm (w_\infty^2 + 4\sigma)^{\frac{1}{2}}, \tag{A 5a}$$

$$2m = w_\infty \pm (w_\infty^2 + 4[\sigma \pm 2\gamma])^{\frac{1}{2}}. \tag{A 5b}$$

The first of these equations corresponds to (A 4c) and leads to the continuous spectrum $\sigma_r < -\sigma_i^2/w_\infty^2$ found in §4. However, (A 5a,b) lead to the continuous spectra,

$$\sigma_r \leqslant \pm 2\gamma - \frac{\sigma_i^2}{w_\infty^2}. \tag{A 6a, b}$$

The positive root corresponds to the case when $\tilde{u} = -\gamma\tilde{U}$ in (A 4a,b) and, surprisingly, we obtain an unstable continuous spectrum for $\gamma > 0$. However, it remains to be seen whether this unstable continuous spectrum can induce a physically relevant exponentially growing solution. In order to answer this question we shall now seek a large-time solution of (A 2); the structure we choose is based on the assumption that at large times the unstable spectrum (A 6a) effectively dominates the flow. The first step in our solution procedure is the substitution

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{U}) = (u^+, v^+, w^+, U^+)\exp\{\tfrac{1}{2}w_\infty \zeta - \tfrac{1}{4}w_\infty^2 T + 2\gamma T\}. \tag{A 7}$$

The functions u^+, v^+ etc., are then found in the regions where $\zeta = O(1)$, $\zeta = O(T^{\frac{1}{2}})$. In the region where $\zeta = O(1)$ we write

$$(u^+, v^+, w^+, U^+) = T^j(u_0^+ v_0^+, w_0^+, U_0^+) + \dots,$$

where u_0^+ etc., depend only on ζ. The problem for these functions is obtained from (A 2) and solved subject to

$$u_0^+ = v_0^+ = w_0^+ = U_0^+ = 0, \quad \zeta = 0,$$

and the unknown quantities $u_0^{+\prime}(0), v_0^{+\prime}(0), w_0^{+\prime}$ are chosen such that

$$u_0^+ \sim \gamma\zeta, \quad U_0^+ \sim -\zeta, \quad v_0 \to 0 \text{ exponentially } \zeta \to \infty.$$

The constant j again remains unknown in the solution of the $\zeta = O(1)$ problem; however an investigation of the problem with $\zeta = O(T^{\frac{1}{2}})$ shows that matching with the above solution can only be achieved if j is again determined by (4.12). Thus we conclude that the steady-state solutions with $w_\infty > 8\gamma$ are linearly unstable and so not likely to be observed.

REFERENCES

Balakumar, P., Hall, P. & Malik, M. 1991 The nonparallel receptivity for rotating disc flows. *Theor. Comp. Fluid Dyn.* (to appear).

Banks, W. H. H. & Zaturska, A. B. 1979 The collision of unsteady laminar boundary layers. *J. Engng Maths* **13**, 193.

Banks, W. H. H. & Zaturska, A. B. 1989 Eigenvalues at a three-dimensional stagnation point. *Acta Mechanica* **78**, 39.

BASSOM, A. P. & GAJJAR, J. S. B. 1988 Non-stationary cross-flow vortices in three-dimensional boundary layer flows. *Proc. R. Soc. Lond.* A **417**, 179.

BODONYI, R. J. & NG, B. 1984 On the stability of the similarity solution for swirling flow above an infinite rotating disc. *J. Fluid Mech.* **144**, 311.

BODONYI, R. J. & STEWARTSON, K. 1977 The unsteady laminar boundary layer on a disc in counter-rotating fluids. *J. Fluid Mech.* **79**, 669.

DAVEY, A. 1961 Boundary-layer flow at a saddle point of attachment. *J. Fluid Mech.* **4**, 593.

FEDERVOV, B. I., PLAVNIK, G. Z., PROKHOLOV, I. V. & ZHUKHOVITSKII, L. G. 1976 Transitional flow conditions on a rotating disc. *J. Engng Phys.* **31**, 1448.

GREGORY, N., STUART, J. T. & WALKER, W. S. 1955 On the stability of three-dimensional boundary layers with applications to the flow due to a rotating disc. *Phil. Trans. R. Soc. Lond.* A **248**, 155.

HALL, P. 1986 An asymptotic investigation of the stationary modes of instability of the boundary layer on a rotating disc. *Proc. R. Soc. Lond.* A **406**, 93.

HALL, P. & SMITH, F. T. 1991 On strongly nonlinear vortex/wave interactions in boundary layer transition. *J. Fluid Mech.* **227**, 641–666.

HOWARTH, L. 1951 The boundary layer in three-dimensional flow. Part II. The stagnation point. *Phil. Mag.* **7** (62), 1433.

KÁRMÁN, T. 1921 Über laminare und turbulente reibung. *Z. angew. Math. Mech.* **1**, 233.

MACKERRELL, S. O. 1987 A nonlinear asymptotic investigation of the three-dimensional boundary layer on a rotating disc. *Proc. R. Soc. Lond.* A **413**, 497.

MALIK, M. 1986 The neutral curve for stationary disturbances in rotating disc flow. *J. Fluid Mech.* **164**, 275.

SCHOFIELD, D. & DAVEY, A. 1967 Dual solutions of the boundary layer of attachment. *J. Fluid Mech.* **30**, 809.

STUART, J. T. 1988 Nonlinear Euler partial differential equations: singularities in their solution. In *Symposium to Honour C. C. Lin*. World Science.